Jumping coefficients and spectrum of a hyperplane arrangement

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Abstract. In an earlier version of this paper written by the second named author, we showed that the jumping coefficients of a hyperplane arrangement depend only on the combinatorial data of the arrangement as conjectured by Mustaţă. For this we proved a similar assertion on the spectrum. After this first proof was written, the first named author found a more conceptual proof using the Hirzebruch-Riemann-Roch theorem where the assertion on the jumping numbers was proved without reducing to that for the spectrum. In this paper we improve these methods and show that the jumping numbers and the spectrum are calculable in low dimensions without using a computer. In the reduced case we show that these depend only on fewer combinatorial data, and give completely explicit combinatorial formulas for the jumping coefficients and (part of) the spectrum in the case the ambient dimension is 3 or 4. We also give an analogue of Mustaţă's formula for the spectrum.

Introduction

This paper combines and improves two unpublished preprints: [29] which gave the first proof of Theorem 1 below, and [5] which gave a formula for the spectrum using the Hirzebruch-Riemann-Roch theorem [19] together with the combinatorial description of the cohomology ring of the wonderful model [8].

Let D be a hyperplane arrangement in $X = \mathbf{C}^n$ with $D_i\,(i \in \Lambda)$ the irreducible components. In this paper we assume D is central and essential (i.e. $0 \in D_i$ for any i, and $\bigcap_i D_i = \{0\}$) since the multiplier ideals and the spectrum are defined locally. We also assume that D is not a divisor with normal crossings (i.e. $\deg D_{\mathrm{red}} > n$). However, we do *not* assume D is *reduced*.

For a positive rational number α, the multiplier ideal $\mathcal{J}(X, \alpha D)$ is a coherent ideal of the structure sheaf $\mathcal{O}_X$ defined by the local integrability of $|g|^2/|f|^{2\alpha}$ for $g \in \mathcal{O}_X$, where f is a defining polynomial of D, see [21], [23]. This gives a decreasing sequence of ideals $\mathcal{J}(X, \alpha D)$ which is locally constant outside a locally finite subset $\mathrm{JC}(D)$ of $\mathbf{Q}$. The elements of $\mathrm{JC}(D)$ are called the *jumping coefficients*. Here we may restrict to the intersection with the interval $(0, 1)$ since $1 \in \mathrm{JC}(D)$ and for $\alpha > 0$, we have $\alpha \in \mathrm{JC}(D)$ if and only if $\alpha + 1 \in \mathrm{JC}(D)$. A formula for the multiplier ideals of a hyperplane arrangement

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D was obtained by Mustaţă [22]. It was conjectured there that $\mathrm{JC}(D)$ depends only on the combinatorial data of a hyperplane arrangement.

For simplicity, consider the case where $n = 3$, D is reduced, and $\mathrm{mult}_x D \le 3$ for any $x \ne 0$. Let $D^{\mathrm{nnc}} \subset D$ denote the complement of the subset consisting of normal crossing singularities. This is the union of the lines of multiplicity 3, and corresponds to a finite subset $\mathbf{P}(D^{\mathrm{nnc}}) \subset \mathbf{P}^2$. Let $\mathbf{C}[x, y, z]_i$ denote the vector space of homogeneous polynomials of degree i. Set $d = \deg D$. In this case Mustaţă's formula implies the following (see [22], Cor. 2.1): If $\alpha = j/d \in (\frac{2}{3}, 1)$ with $j \in \mathbf{Z}$, then

$$\alpha \in \mathrm{JC}(D) \iff \exists\, g \in \mathbf{C}[x, y, z]_{j-3} \setminus \{0\} \text{ with } g^{-1}(0) \supset \mathbf{P}(D^{\mathrm{nnc}}). \tag{0.1}$$

The last condition may a priori depend on the position of $\mathbf{P}(D^{\mathrm{nnc}})$, and the conjecture is rather nontrivial. It turns out, however, that the points of $\mathbf{P}(D^{\mathrm{nnc}})$ are always in a generic position as far as the above condition is concerned, see a remark on the surjectivity of (0.2) after Theorem 5 below. Note that there is an example such that $\frac{j}{d} \notin \mathrm{JC}(D)$, see Example (3.6) below. (This implies a negative answer to [22], Question 2.7 even in the case $j = 5$.)

Using an inductive argument, we gave in [29] the first proof of the following.

Theorem 1. *The jumping coefficients and the spectrum of a hyperplane arrangement depend only on the combinatorial data.*

The spectrum $\mathrm{Sp}(f)$ of a hypersurface singularity was defined by J. Steenbrink [31] using the monodromy and the Hodge filtration on the Milnor cohomology where f is a function defining locally D. The spectrum $\mathrm{Sp}(f)$ is a fractional polynomial $\sum_\alpha n_{f,\alpha} t^\alpha$ where $n_{f,\alpha} = 0$ for $\alpha \notin \mathbf{Q} \cap (0, n)$ (see [6], Prop. 5.2), and α is called a nontrivial exponent if $n_{f,\alpha} \ne 0$. It was shown in [3] that, for $\alpha \in (0, 1)$, it is a jumping coefficient if it is a nontrivial exponent, and the converse holds in the isolated singularity case. Using [6], we can show for any holomorphic function f that the converse also holds if α is an isolated jumping coefficient (i.e. if α is not a jumping coefficient for $D \setminus \{0\}$), see Proposition (4.2) below.

Returning to the case of an essential central hyperplane arrangement, let g_i be the defining linear function of D_i with m_i the multiplicity. Let V be an intersection of D_i which is called an *edge*. Set $f_{X/V} = \prod_{D_i \supset V} g_i^{m_i}$. This is viewed as a function on X/V, and defines a hyperplane arrangement $D_V \subset X/V$. Let $D^{\mathrm{nrnc}} \subset D$ denote the complement of the subset consisting of reduced normal crossing singularities. By Proposition (4.2) we then get the following.

Proposition 1. *For $\alpha \in (0, 1)$, we have $\alpha \in \mathrm{JC}(D)$ if and only if there is an edge $V \subset D^{\mathrm{nrnc}}$ with $n_{f_{X/V},\alpha} \ne 0$.*

So the assertion of Theorem 1 for the jumping coefficients is reduced to the corresponding assertion for the spectrum since the combinatorial data of $D_V \subset X/V$ are obtained by restricting those of D. The spectrum of a hyperplane arrangement is calculated by using the canonical embedded resolution together with the filtered logarithmic complexes associated to certain local systems as in [16]. The assertion is then reduced to the calculation of the restriction of the de Rham complex to the exceptional divisors of the canonical

embedded resolution where we need some arguments as in [8]. Note that the moduli space of hyperplane arrangements with fixed combinatorial data is not necessarily connected as shown in [25] as a corollary of a deep theorem (see also [27], 5.7 for a direct argument), and hence we cannot prove Theorem 1 by simply using a deformation argument.

After an earlier version of the above proof of Theorem 1 due to the second named author [29] was written, the first named author ([4], [5]) found a more conceptual proof using the Hirzebruch-Riemann-Roch theorem and a combinatorial description [8] of the cohomology ring of the canonical embedded resolution of the corresponding hyperplane arrangement in $\mathbf{P}^{n-1}$. The new proof implies formulas for the jumping coefficients and the spectrum in terms of the combinatorial data where the assertion for the jumping numbers is proved without using Proposition 1 (although it needs a compactification of $X = \mathbf{C}^n$ so that the calculation becomes more complicated than the proof using Proposition 1). It is also possible to write down a formula for the spectrum as in [5] by summarizing the arguments in (5.3–4) in this paper. (Note that [4] did not deal with the whole spectrum.) Stimulated by this new proof, there was an improvement of the inductive argument. Using these we can prove Theorem 2 below.

Let $\mathcal{S}(D)^{\rm nnc}$ denote the set of edges V contained in $D^{\rm nnc}$. Let $\subset$, $\mu(V)$, and $\gamma(V)$ respectively denote the inclusion relation, multiplicity of D along V, and the codimension of V, see (1.1) below. Then we have the following.

Theorem 2. *Assume D is reduced. Then the jumping coefficients and the coefficients $n_{f,\alpha}$ of the spectrum for $\alpha \in (0,1] \cup (n-1,n)$ depend only on the weak combinatorial equivalence class, i.e. on the set $\mathcal{S}(D)^{\rm nnc}$ together with the combinatorial data $\subset$, μ, γ. For $\alpha \in (1,n-1]$, $n_{f,\alpha}$ depends only on $\mathcal{S}(D)^{\rm nnc}$ together with $\subset$, μ, γ and also the subsets $\mathcal{S}^{D_i} := \{V \in \mathcal{S}(D)^{\rm nnc} \mid V \subset D_i\}$ $(i \in \Lambda)$.*

The weak combinatorial equivalence is strictly weaker than the usual combinatorial equivalence. Indeed, if $n = 3$, the former is determined only by $d = \deg D$ and

$$\nu_m^{(2)} = \#\{y \in \mathbf{P}(D) \mid \mathrm{mult}_y \mathbf{P}(D) = m\} \quad \text{for } m \geq 3,$$

where $\mathbf{P}(D) \subset \mathbf{P}^2$, see also (1.1.3). For instance, in the case where $d = 7$, $\nu_3^{(2)} = 3$, and $\nu_m^{(2)} = 0\,(m > 3)$, there are two possibilities of combinatorial data depending on whether the three points of multiplicity 3 are on a same line or not.

In this paper we also show that the jumping coefficients and the spectrum of hyperplane arrangements are calculable in low dimensions without using a computer as in [4], [5] (although the formula is rather complicated). This was partly made possible by restricting the centers of the blow-ups to those contained in $D^{\rm nnc}$. For instance we have the following.

Theorem 3. *Assume D reduced and $n = 3$. Then $n_{f,\alpha} = 0$ if $\alpha d \notin \mathbf{Z}$, and we have for $\alpha = \frac{i}{d} \in (0,1]$ with $i \in [1,d] \cap \mathbf{Z}$*

$$\begin{aligned}
n_{f,\alpha} &= \tbinom{i-1}{2} - \textstyle\sum_m \nu_m^{(2)} \tbinom{\lceil im/d \rceil - 1}{2}, \\
n_{f,\alpha+1} &= (i-1)(d-i-1) - \textstyle\sum_m \nu_m^{(2)} \big(\lceil im/d \rceil - 1\big)\big(m - \lceil im/d \rceil\big), \\
n_{f,\alpha+2} &= \tbinom{d-i-1}{2} - \textstyle\sum_m \nu_m^{(2)} \tbinom{m - \lceil im/d \rceil}{2} - \delta_{i,d},
\end{aligned}$$

where $\lceil\beta\rceil := \min\{k\in\mathbf{Z}\mid k\ge\beta\}$*, and* $\delta_{i,d}=1$ *if* $i=d$ *and* 0 *otherwise.*

Theorem 4. *Assume* D *reduced and* $n=4$*. Let* $\nu_m^{(2)}$, $\nu_{m'}^{(3)}$, $\nu_{m,m'}^{(2,3)}$ *be as in* (1.1.3) *below. Then* $n_{f,\alpha}=0$ *for* $\alpha d\notin\mathbf{Z}$*, and we have for* $\alpha=\frac{i}{d}\in(0,1]$ *with* $i\in[1,d]\cap\mathbf{Z}$

$$\begin{aligned}n_{f,\alpha}=\tbinom{i-1}{3}-\textstyle\sum_{m,m'}\nu_{m,m'}^{(2,3)}\Big(2\tbinom{\lceil im/d\rceil-1}{3}-\tbinom{\lceil im/d\rceil-1}{2}\big(\lceil im'/d\rceil-3\big)\Big)\\ +\textstyle\sum_m\nu_m^{(2)}\Big(2\tbinom{\lceil im/d\rceil-1}{3}-(i-3)\tbinom{\lceil im/d\rceil-1}{2}\Big)-\textstyle\sum_{m'}\nu_{m'}^{(3)}\tbinom{\lceil im'/d\rceil-1}{3}.\end{aligned}$$

The formula is similar for $n=4$ and $\alpha\in(3,4)$. However, it requires some more combinatorial data, and is more complicated for $n=4$ and $\alpha\in(1,3]$. Those are left to the reader as exercises. Note that the formula for $n=3$ and $\alpha\in(0,1]$ is quite similar to a formula for the Hodge number in [15], Th. 6. As for the jumping coefficients, it is well-known that $\mathrm{JC}(D)\cap(0,1)=\{i/d\mid i\in\mathbf{Z}\cap[2,d)\}$ with $d=\deg D$ if $n=2$ and D is reduced. Combined with Proposition 1, Theorems 3–4 then imply the following.

Corollary 1. *Assume* D *reduced and* $n=3$*. Then* $\alpha\in(0,1)$ *is a jumping coefficient of* D *if and only if there is* $m\in\mathbf{N}\cap[3,\infty)$ *such that* $m\alpha\in\mathbf{Z}\cap[2,m)$ *and* $\nu_m^{(2)}\ne0$ *or there is* $i\in\mathbf{Z}\cap[3,d)$ *such that* $\alpha=\frac{i}{d}$ *and* $n_{f,\alpha}\ne0$ *in Theorem* 3*.*

Corollary 2. *Assume* D *reduced and* $n=4$*. In the notation of* (1.1.3)*,* $\alpha\in(0,1)$ *is a jumping coefficient of* D *if and only if there is* $m\in\mathbf{N}\cap[3,\infty)$ *such that* $m\alpha\in\mathbf{Z}\cap[2,m)$ *and* $\nu_m^{(2)}\ne0$*, or there is* $V\in\mathcal{S}(D)^{\mathrm{nnc}}$ *together with* $i\in\mathbf{Z}\cap[3,\mu(V))$ *such that* $\operatorname{codim}V=3$*,* $\alpha=i/\mu(V)$ *and* $n_{f_{X/V},\alpha}\ne0$ *in Theorem* 3*, or there is* $i\in\mathbf{Z}\cap[4,d)$ *such that* $\alpha=i/d$ *and* $n_{f,\alpha}\ne0$ *in Theorem* 4*. Here* $\mu(V)=\mathrm{mult}_VD$*.*

We have a formula for the spectrum analogous to Mustaţă's formula [22] as follows.

Theorem 5. *With the notation of* (1.1) *below, let* $\mathcal{I}_V\subset\mathbf{C}[X]$ *denote the reduced ideal of* $V\in\mathcal{S}':=\mathcal{S}(D)^{\mathrm{nrnc}}$*. For* $\alpha=\frac{j}{d}\in(0,1]$ *with* $j\in[1,d]\cap\mathbf{Z}$ *we have*

$$n_{f,\alpha}=\dim\Big(\textstyle\bigcap_{V\in\mathcal{S}'\setminus\{0\}}\mathcal{I}_V^{\lceil\alpha\mu(V)\rceil-\gamma(V)}\cap\mathbf{C}[X]_{j-n}\Big).$$

By Proposition (4.2) below this is compatible with Mustaţă's formula [22] for $\alpha\in(0,1)$ in the case $\mu(V)\alpha\notin\mathbf{Z}$ for any $V\in\mathcal{S}(D)\setminus\{0\}$, see also (3.3) below. Here $\mathbf{C}[X]_{j-n}$ denotes the space of homogeneous polynomials of degree $j-n$, and the intersection with $\mathcal{I}_V^{e_V}$ gives a restriction condition for $g\in\mathbf{C}[X]_{j-n}$ as in the right-hand side of (0.1) where $e_V=\lceil\alpha\mu(V)\rceil-\gamma(V)$. In the case $n=3$ and D is reduced, this restriction condition is given by the condition that $g\in\mathbf{m}_V^{e_V}$, where g is viewed as a section of $\mathcal{O}_{\mathbf{P}^2}(j-3)$ and $\mathbf{m}_V\subset\mathcal{O}_{\mathbf{P}^2,y}$ is the maximal ideal with y the closed point of $\mathbf{P}^2$ corresponding to V. (Here $\mathcal{O}_{\mathbf{P}^2}(j-3)$ is locally trivialized.) Since $\dim\mathcal{O}_{\mathbf{P}^2,y}/\mathbf{m}_V^{e_V}=\binom{e_V+1}{2}$, the first equality of Theorem 3 for $\alpha\in(0,1]$ implies the surjectivity of

$$H^0(\mathbf{P}^2,\mathcal{O}_{\mathbf{P}^2}(j-3))\to\textstyle\bigoplus_{V\in\mathcal{S}'\setminus\{0\}}\mathcal{O}_{\mathbf{P}^2,y}/\mathbf{m}_V^{e_V},\tag{0.2}$$

which means that the above restriction conditions for $V\in\mathcal{S}'\setminus\{0\}$ are always independent so that they give the maximal restriction condition in total. This is closely related to

the non-degeneracy of the matrix after (3.6.1). Note also that the above surjectivity is equivalent to the vanishing

$$H^1(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(i-3)\otimes_{\mathcal{O}}\mathcal{I}(\alpha))=0,$$

where $\mathcal{I}(\alpha) := \mathrm{Ker}(\mathcal{O}_{\mathbf{P}^2}\to\bigoplus_{V\in\mathcal{S}'\setminus\{0\}}\mathcal{O}_{\mathbf{P}^2,y}/\mathbf{m}_V^{e_V})$.

It is known that the jumping coefficients are closely related to the roots of the b-function, see [14] and (3.5) below. For the moment it is unclear whether an analogue of Theorem 1 holds for the b-function. As for the relation with topology, note that the spectrum does not determine each Betti number of the Milnor fiber of an hyperplane arrangement, see [7], Ex. 5.4–5 and [12], p. 213, Ex. 4.16.

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In Section 1, we review some facts related to hyperplane arrangements, spectrum and jumping coefficients. In Section 2, we essentially reproduce Section 2 of [29] on the canonical embedded resolution of a projective hyperplane arrangement, see also [8]. In Section 3, we prove Theorems 3 and 5. In Section 4, we give an improved version of the first proof of Theorem 1 together with proofs of Theorems 2–4 by induction. In Section 5, we improve some arguments in [4], [5] (using D^{nnc}), and give proofs of Theorems 1–4 using the Hirzebruch-Riemann-Roch theorem and the combinatorial description of the cohomology ring of the embedded resolution.

1. Preliminaries

1.1. Hyperplane arrangements. Let D be a central hyperplane arrangement in $X=\mathbf{C}^n$ with D_i ($i\in\Lambda$) the irreducible components of D, where central means that all the D_i pass through the origin, see [24]. We also assume D is essential, and is not a divisor with normal crossings, i.e. $\bigcap_i D_i=\{0\}$ and $\deg D_{\mathrm{red}}>n$. We define a set of vector subspaces of X by

$$\mathcal{S}(D)=\{\textstyle\bigcap_{i\in I}D_i\}_{I\subset\Lambda,I\ne\emptyset},$$

where I runs over the nonempty subsets of Λ. (Note that we may have $\bigcap_{i\in I}D_i=\bigcap_{i\in I'}D_i$ with $I\ne I'$.) For $V\in\mathcal{S}(D)$, define

$$I(V)=\{i\in\Lambda\mid D_i\supset V\},$$

so that $V=\bigcap_{i\in I(V)}V_i$. There are functions $\gamma,\mu,\mu_{\mathrm{red}}:\mathcal{S}(D)\to\mathbf{N}$ such that for $V\in\mathcal{S}(D)$

$$\begin{aligned}\gamma(V)&:=\mathrm{codim}_X V=\min\{|I|\;\big|\;\textstyle\bigcap_{i\in I}D_i=V\},\\ \mu(V)&:=\mathrm{mult}_V D=\textstyle\sum_{i\in I(V)}\mu(D_i),\\ \mu_{\mathrm{red}}(V)&:=\mathrm{mult}_V D_{\mathrm{red}}=\#I(V).\end{aligned}\tag{1.1.1}$$

There is a natural order on $\mathcal{S}(D)$ defined by the inclusion relation $\subset$.

Let $D^{\mathrm{nnc}} \subset D$ denote the complement of the subset consisting of normal crossing singularities. Here "normal crossing" means that the associated reduced variety has normal crossings. We define similarly D^{nrnc} by replacing "normal crossing" with "reduced normal crossing" so that $D^{\mathrm{nnc}} \subset D^{\mathrm{nrnc}}$. (In this paper we do not assume $D^{\mathrm{nnc}} \neq \emptyset$.) Set

$$\mathcal{S}(D)^{\mathrm{nnc}} = \{V \in \mathcal{S}(D) \mid V \subset D^{\mathrm{nnc}}\} \ \text{(similarly for } \mathcal{S}(D)^{\mathrm{nrnc}}). \tag{1.1.2}$$

In this paper D^{nrnc} is used only in Theorem 5 and Mustaţă's formula, see (3.3–4).

For $\mathcal{S} := \mathcal{S}(D)^{\mathrm{nnc}}$, set

$$\begin{aligned}
&\mathcal{S}^{D_i} = \{V \in \mathcal{S} \mid V \subset D_i\}, \quad \mathcal{S}^{(i)} = \{V \in \mathcal{S} \mid \gamma(V) = i\}, \\
&\nu_m^{(i)} = \#\{V \in \mathcal{S}^{(i)} \mid \mu(V) = m\}, \\
&\nu_{m,m'}^{(i,j)} = \#\{(V, V') \in \mathcal{S}^{(i)} \times \mathcal{S}^{(j)} \mid V \supset V',\, \mu(V) = m,\, \mu(V') = m'\}.
\end{aligned} \tag{1.1.3}$$

This is compatible with the definition of $\nu_m^{(2)}$ in Introduction.

1.2. Combinatorial equivalence class. With the above notation we call

$$\mathcal{S}(D),\, \subset,\, \mu$$

the (strong) combinatorial data of a hyperplane arrangement D. Note that γ is determined by the inclusion relation $\subset$, and so is μ in the case D is reduced. We call

$$\mathcal{S}(D)^{\mathrm{nnc}},\, \subset,\, \mu,\, \gamma,$$

the *weak* combinatorial data. We say that two hyperplane arrangements D and D' are combinatorially equivalent if there is a one-to-one correspondence between $\mathcal{S}(D)$ and $\mathcal{S}(D')$ in a compatible way with $\subset, \mu$ (similarly for weak combinatorial equivalence).

1.3. Milnor fiber and the covering. In this subsection $D \subset X := \mathbf{C}^n$ is defined by a homogeneous polynomial f. Set $Z = \mathbf{P}(D)$. This is a subvariety of $Y := \mathbf{P}^{n-1}$ defined by f. There is a ramified covering

$$Y' := \mathcal{S}pec_Y(\textstyle\bigoplus_{0 \le k < d} \mathcal{S}^k) \xrightarrow{\pi} Y,$$

where $\mathcal{S}^k = \mathcal{O}_Y(-k)$ and f induces morphisms $\mathcal{O}_Y(-k-d) \to \mathcal{O}_Y(-k)$ defining a ring structure on $\bigoplus_{0 \le k < d} \mathcal{S}^k$.

Let $U = Y \setminus Z$, and U' be the restriction of Y' over U. Then U' is étale over U, and the Milnor fiber $f^{-1}(1)$ is identified with U'. Indeed, Y' is identified with a section of the line bundle over Y corresponding to $\mathcal{O}_Y(1)$, and U' is identified with a section of its dual bundle which is isomorphic to the blow-up of $\mathbf{C}^n$ at the origin. So U' is identified with the divisor on $\mathbf{C}^n$ defined by $f = 1$.

The geometric Milnor monodromy is induced by an automorphism of $\mathbf{C}^n$ defined by

$$T_g : (x_1, \ldots, x_n) \mapsto (\xi x_1, \ldots, \xi x_n),$$

where $\xi = \exp(2\pi\sqrt{-1}/d)$. Note that the monodromy of the cohomology local system associated with the Milnor fibration on a punctured disk is given by $(T_g^*)^{-1}$. It is well-known (see e.g. [7], [12]) that

$$\mathcal{S}^k = \mathrm{Ker}((T_g^*)^{-1} - \xi^k) \subset \pi_*\mathcal{O}_{Y'} = \textstyle\bigoplus_{0\le k<d} \mathcal{S}^k. \tag{1.3.1}$$

For the convenience of the reader, we include here a proof. Using the projective coordinates $z_0, \dots, z_n$ of $\mathbf{P}^n \supset \mathbf{C}^n$ such that $x_i = z_i/z_0$ for $i \in [1, n]$, the geometric monodromy is induced by

$$T_g : (z_0, z_1, \dots, z_n) \mapsto (\xi^{-1}z_0, z_1, \dots, z_n).$$

After changing the order of the coordinates $z_1, \dots, z_n$ if necessary, let $y_i = z_i/z_n$ on $\{z_n \ne 0\} \subset \mathbf{P}^n$ for $i \in [0, n-1]$. Set $h(y_0, \dots, y_{n-1}) = f(z_1, \dots, z_n)/z_n^d$. Then $Y' \subset \mathbf{P}^n$ is locally defined by the equations

$$f(x_1, \dots, x_n) = 1, \quad f(z_1, \dots, z_n) = z_0^d, \quad h(y_0, \dots, y_{n-1}) = y_0^d,$$

and the restriction of $\pi_*\mathcal{O}_{Y'}$ to $\mathbf{C}^{n-1} = \{z_n \ne 0\} \subset \mathbf{P}^{n-1}$ is identified with

$$\pi_*(\mathcal{O}_{\mathbf{C}^n}/(y_0^d - h(y_0, \dots, y_{n-1})) = \textstyle\bigoplus_{k=0}^{d-1} \mathcal{O}_{\mathbf{C}^{n-1}} y_0^k,$$

where $\{z_n \ne 0\} \subset \mathbf{P}^n$ is identified with $\mathbf{C}^n$. On the other hand, the action of T_g^* on the coordinate y_i is the multiplication by ξ^{-1} for $i = 0$, and the identity for $i \ne 0$. So (1.3.1) follows.

1.4. Spectrum. With the notation of (1.3) the spectrum $\mathrm{Sp}(f) = \sum_{\alpha\in\mathbf{Q}} n_{f,\alpha} t^\alpha$ is defined by

$$n_{f,\alpha} = \textstyle\sum_j (-1)^{j-n+1} \dim \mathrm{Gr}_F^p \widetilde{H}^j(f^{-1}(1), \mathbf{C})_\lambda$$
$$\text{with } p = \lfloor n - \alpha \rfloor, \ \lambda = \exp(-2\pi i\alpha),$$

where $\widetilde{H}^j(f^{-1}(1), \mathbf{C})_\lambda$ is the λ-eigenspace of the reduced cohomology of $f^{-1}(1) \subset \mathbf{C}^n$ for the semi-simple part of the Milnor monodromy, and F is the Hodge filtration, see [31]. Here $\lfloor\beta\rfloor := \max\{k \in \mathbf{Z} \mid k \le \beta\}$. By [6], Prop. 5.2, we have

$$n_{f,\alpha} = 0 \quad \text{if} \quad \alpha \notin (0, n).$$

By (1.3.1) $\mathcal{S}^k$ has a meromorphic connection with regular singularities along Z, and hence the localization $\mathcal{S}^k(*Z)$ along Z is a regular holonomic $\mathcal{D}_Y$-module, which is locally isomorphic to $\mathcal{O}_Y(*Z)h^{k/d}$ where h is as in (1.3). (This is proved by using the equation $h = y_0^d$.) Moreover we get by (1.3.1)

$$H^j(U, \mathrm{DR}(\mathcal{S}^k|_U)) = H^j(f^{-1}(1), \mathbf{C})_\lambda \quad \text{with } \lambda = \exp(2\pi ik/d). \tag{1.4.1}$$

Since $\mathcal{S}^k$ has rank 1, this implies

$$\textstyle\sum_{j\in\mathbf{Z}} (-1)^j \dim H^j(f^{-1}(1), \mathbf{C})_\lambda = \chi(U). \tag{1.4.2}$$

Let $\rho : \widetilde{Y} \to Y$ be an embedded resolution of Z inducing an isomorphism over $Y \setminus Z$. We have a divisor with normal crossings on $\widetilde{Y}$

$$\widetilde{Z} := \rho^*Z = Z' + Z'' \quad \text{with } Z' = \textstyle\sum_{j\in J'} m_jE_j, \;\; Z'' = \textstyle\sum_{j\in J''} m_jE_j,$$

where Z' is the proper transform of Z, Z'' is the exceptional divisor, and the E_j are the irreducible components with multiplicity m_j. Set $J = J' \cup J''$.

Let $\widetilde{\mathcal{S}}^k$ be the Deligne extension of $\mathcal{S}^k|_U$ over $\widetilde{Y}$ such that the eigenvalues of the residue of the connection are contained in $[0, 1)$, see [9]. Let $\widetilde{H}$ be the total (or proper) transform of a general hyperplane H of Y. Using the pull-back of the above local form $\mathcal{O}_Y(*Z)h^{k/d}$, we get

$$\widetilde{\mathcal{S}}^k = \mathcal{O}_{\widetilde{Y}}(-k\widetilde{H} + \textstyle\sum_{j\in J} \lfloor km_j/d\rfloor E_j),$$

since $\mathcal{O}_Y(Z) = \mathcal{O}_Y(dH)$. Indeed, the eigenvalue of the residue of the connection along E_j is

$$km_j/d - \lfloor km_j/d\rfloor.$$

Note that the above summation is taken over J'' in case D is reduced since $\lfloor km_j/d\rfloor = 0$ if $m_j = 1$ and $k \in [0, d)$. It is known (see e.g. [16]) that the associated filtered logarithmic complex together with the filtration σ (see [10]) calculates the Hodge filtration on the cohomology. It coincides with the Hodge filtration obtained from the theory of mixed Hodge modules. (This is shown by using e.g. [26], 3.11.) So we get

$$\mathrm{Gr}_F^p H^{p+q}(f^{-1}(1), \mathbf{C})_\lambda = H^q(\widetilde{Y}, \Omega^p_{\widetilde{Y}}(\log \widetilde{Z}) \otimes_{\mathcal{O}} \mathcal{O}_{\widetilde{Y}}(-k\widetilde{H} + \textstyle\sum_j \lfloor km_j/d\rfloor E_j)). \tag{1.4.3}$$

Since $f^{-1}(1)$ is affine and $(n-1)$-dimensional, we have for $q > n-1-p$

$$H^q(\widetilde{Y}, \Omega^p_{\widetilde{Y}}(\log \widetilde{Z}) \otimes_{\mathcal{O}} \mathcal{O}_{\widetilde{Y}}(-k\widetilde{H} + \textstyle\sum_{j\in J} \lfloor km_j/d\rfloor E_j)) = 0. \tag{1.4.4}$$

This is closely related to the Kodaira-Nakano type vanishing theorem in [16]. We also have

$$\begin{aligned} \Omega^{n-1}_{\widetilde{Y}}(\log \widetilde{Z}) &= \mathcal{O}_{\widetilde{Y}}(-n\widetilde{H} + \textstyle\sum_{j\in J} c_jE_j), \\ &= \mathcal{O}_{\widetilde{Y}}((d-n)\widetilde{H} + \textstyle\sum_{j\in J} (c_j - m_j)E_j), \end{aligned} \tag{1.4.5}$$

where c_j is the codimension of the center of the blow-up corresponding to E_j.

From (1.4.3) we deduce

1.5. Proposition. *For $\alpha = \frac{i}{d} + \ell$ with $i = d - k \in [1, d]$ and $\ell = n - 1 - p \in [0, n-1]$*

$$n_{f,\alpha} = (-1)^\ell \chi(\widetilde{Y}, \Omega^p_{\widetilde{Y}}(\log \widetilde{Z}) \otimes_{\mathcal{O}} \mathcal{O}_{\widetilde{Y}}(-k\widetilde{H} + \textstyle\sum_{j\in J} \lfloor km_j/d\rfloor E_j)). \tag{1.5.1}$$

Here $\lfloor\beta\rfloor := \max\{k \in \mathbf{Z} \mid k \le \beta\}$.

1.6. Compatibility with the usual definition. The above definition of the spectrum coincides with the definition using the mixed Hodge structure obtained by $H^k i_0^*\psi_f\mathbf{Q}_X$

where ψ_f denotes the nearby cycle functor (see [11]) and $i_0 : \{0\} \to X_0 := f^{-1}(0)$ denotes the inclusion. Indeed, by the compatibility of ψ with the direct image under the blow-up $\rho : \widetilde{X} \to X$ at 0 (see [26], 2.14), we have isomorphisms of mixed Hodge modules

$$ {}^pR^k \rho_* \psi_{\widetilde{f}}(\mathbf{Q}_{\widetilde{X}}[n]) = \begin{cases} \psi_f(\mathbf{Q}_X[n]) & \text{if } k = 0, \\ 0 & \text{if } k \neq 0, \end{cases} $$

where ψ is shifted by -1 so that it preserves the perverse sheaves. So we get an isomorphism in the derived category of (algebraic) mixed Hodge modules

$$ \psi_f \mathbf{Q}_X = \mathbf{R}\rho_* \psi_{\widetilde{f}} \mathbf{Q}_{\widetilde{X}}. $$

Note that Y is identified with the exceptional divisor of the blow-up. Let $i_Y : Y \to \widetilde{X}_0 := \widetilde{f}^{-1}(0)$ and $a_Y : Y \to pt$ denote the natural morphisms. Using the base change of ρ by i_0, we then get

$$ H^k i_0^* \psi_f \mathbf{Q}_X = H^k i_0^* \mathbf{R}\rho_* \psi_{\widetilde{f}} \mathbf{Q}_{\widetilde{X}} = H^k (a_Y)_* i_Y^* \psi_{\widetilde{f}} \mathbf{Q}_{\widetilde{X}}. $$

Moreover, if $j' : U \to \widetilde{X}_0$ denotes the inclusion, we have by [6], 4.2

$$ (i_Y)_* i_Y^* \psi_{\widetilde{f}} \mathbf{Q}_{\widetilde{X}} = \mathbf{R} j'_* j'^* \psi_{\widetilde{f}} \mathbf{Q}_{\widetilde{X}}. $$

We get thus

$$ H^k i_0^* \psi_f \mathbf{Q}_X = H^k (a_U)_* j'^* \psi_{\widetilde{f}} \mathbf{Q}_{\widetilde{X}}. $$

So the desired compatibility is reduced to the isomorphism between the λ-eigenspace of the local system $\psi_{\widetilde{f}} \mathbf{C}_{\widetilde{X}}|_U$ and the local system corresponding to $\mathcal{S}^k|_U$ where $\lambda = \exp(2\pi\sqrt{-1}k/d)$ since the Hodge filtration is given by the filtration $\sigma_{\geq p}$ on the logarithmic de Rham complex. To show the isomorphism of local systems, it is enough to show the coincidence of the local monodromies of the two local systems along any irreducible components (indeed, this implies the triviality of the tensor product of one local system with the inverse of the other since a local system of rank 1 on $U \subset \mathbf{P}^{n-1}$ is trivial if the local monodromies are). By the calculation of the residue of the connection in (1.4), the local monodromy of $\mathcal{S}^k|_U$ along E_j is the multiplication by $\exp(-2\pi\sqrt{-1}km_j/d)$ where m_j is the multiplicity of $\widetilde{f}$ along E_j. For $\lambda = \exp(2\pi\sqrt{-1}k/d)$, it is well-known that the monodromy of the λ-eigenspace of the local system $\psi_{\widetilde{f}} \mathbf{C}_{\widetilde{X}}|_U$ along E_j is the multiplication by λ^{-m_j} (since the Milnor fiber is locally defined by $x_0^d x_j^{m_j} = t$ on a neighborhood of a general point of E_j where x_0 defines the exceptional divisor of the blow-up along $0 \in X$), see also [26], 3.3. So the assertion follows.

1.7. Weight filtration on the cohomology of the complement. From now on, we assume that D is an essential central hyperplane arrangement. Let $U = Y \setminus \mathbf{P}(D)$ with $j_U : U \to Y = \mathbf{P}^{n-1}$ the inclusion. Since U is affine, it is known that

$$ (j_U)_! \mathbf{Q}_U[n-1], \quad \mathbf{R}(j_U)_* \mathbf{Q}_U[n-1] $$

are perverse sheaves (see [1]) and underlie mixed Hodge modules (see [26]). So they have the canonical weight filtration W as perverse sheaves. Set

$$\bar{\mathcal{S}}(D) = \mathcal{S}(D) \cup \{X\}, \quad \bar{\mathcal{S}}(D)^{(j)} = \{V \in \bar{\mathcal{S}}(D) \mid \gamma(V) = j\},$$

where $\gamma(V) = \operatorname{codim} V$. Then we can show

$$\begin{aligned} \mathrm{Gr}^W_{n-1-j}((j_U)_!\mathbf{Q}_U[n-1]) &= \textstyle\bigoplus_{V \in \bar{\mathcal{S}}(D)^{(j)}} L_V[n-1-j], \\ \mathrm{Gr}^W_{n-1+j}(\mathbf{R}(j_U)_*\mathbf{Q}_U[n-1]) &= \textstyle\bigoplus_{V \in \bar{\mathcal{S}}(D)^{(j)}} L_V(-j)[n-1-j], \end{aligned} \tag{1.7.1}$$

where the L_V are polarized constant variations of Hodge structures of type $(0,0)$ on $\mathbf{P}(V) \subset \mathbf{P}^{n-1}$. It is enough to show the first isomorphism since the L_V are self-dual by the polarization so that the second follows from the first by duality.

By increasing induction on $j \ge 0$, we define $\mathbf{Q}$-vector spaces L_V $\big(V \in \bar{\mathcal{S}}(D)^{(j)}\big)$ together with morphisms

$$\rho_{V,V'} : L_{V'} \to L_V \quad \text{for } V \in \bar{\mathcal{S}}(D)^{(j)},\ V' \in \bar{\mathcal{S}}(D)^{(j-1)},$$

such that $\rho_{V,V'} = 0$ unless $V \subset V'$ as follows: Set $L_V = \mathbf{Q}$ if $j = 0$ or 1, and $\rho_{V,V'} = id$ if $\gamma(V) = 1$, $\gamma(V') = 0$. Assume L_V and $\rho_{V,V'}$ are defined for $\gamma(V) < j$. For $V \in \bar{\mathcal{S}}(D)^{(j)}$, define

$$\begin{aligned} L_V = \mathrm{Coker}\big(\textstyle\sum \rho_{V',V''} : \bigoplus_{V'' \in \bar{\mathcal{S}}(D)^{j-2}_V} L_{V''} \to \bigoplus_{V' \in \bar{\mathcal{S}}(D)^{j-1}_V} L_{V'}\big), \\ \text{where} \quad \bar{\mathcal{S}}(D)^{(j)}_{V'} := \{V \in \bar{\mathcal{S}}(D) \mid V \supset V'\}. \end{aligned} \tag{1.7.2}$$

The morphism $\rho_{V,V'} : L_{V'} \to L_V$ for $V' \in \bar{\mathcal{S}}(D)^{(j-1)}_V$ is given by the composition

$$L_{V'} \to \textstyle\bigoplus_{V' \in \bar{\mathcal{S}}(D)^{j-1}_V} L_{V'} \to L_V,$$

where the first morphism is the natural inclusion, and the second is the projection to the quotient. Note that the L_V are Hodge structures of type $(0,0)$, and they have a canonical polarization using the semi-simplicity induced by the polarization inductively.

From now on, L_V will be identified with a constant sheaf on $\mathbf{P}(V) \subset Y = \mathbf{P}^{n-1}$ with stalk L_V. Then $\rho_{V',V} : L_{V'} \to L_V$ is viewed as a morphism of sheaves. Define a sheaf on Y by

$$\mathcal{K}^j_Y = \textstyle\bigoplus_{V \in \bar{\mathcal{S}}(D)^{(j)}} L_V.$$

We have the morphism $d : \mathcal{K}^{j-1}_Y \to \mathcal{K}^j_Y$ induced by the $\rho_{V',V}$, and $d \circ d = 0$ by the above construction. Note that the $\mathcal{K}^j_Y[n-1-j]$ and hence $\mathcal{K}^\bullet_Y[n-1]$ are perverse sheaves on Y, see [1]. Then (1.7.1) is reduced to the following lemma since the weight filtration W is induced by the truncation $\sigma_{\ge k}$ up to a shift.

1.8. Lemma. *There is a quasi-isomorphism*

$$(j_U)_!\mathbf{Q}_U \xrightarrow{\sim} \mathcal{K}^\bullet_Y, \tag{1.8.1}$$

induced by the canonical morphism $(j_U)_!\mathbf{Q}_U \to \mathbf{Q}_Y = \mathcal{K}_Y^0$.

Proof. Set
$$Y^{(j)} = \textstyle\bigcup_{V\in\bar{\mathcal{S}}(D)^{(j)}} \mathbf{P}(V), \quad U^{(j)} = Y^{(j)} \setminus Y^{(j+1)}.$$

Let $\sigma_{\le j}\mathcal{K}_Y^\bullet$ denote the quotient complex of $\mathcal{K}_Y^\bullet$ as in [10], 1.4.7, i.e. $(\sigma_{\le j}\mathcal{K}_Y)^i = \mathcal{K}_Y^i$ if $i \le j$, and 0 otherwise. By increasing induction on j we show

(A_j) $\quad C((j_U)_!\mathbf{Q}_U \to \sigma_{\le j}\mathcal{K}_Y^\bullet)[n-2]$ is a perverse sheaf supported on $Y^{(j+1)}$.

This is clear if $j = 0$. Assume (A_{j-1}) holds with $j > 0$. Let y be a general point of $\mathbf{P}(V)$ with $V \in \bar{\mathcal{S}}(D)^{(j)}$. Then (A_{j-1}) implies
$$H^k(\sigma_{<j}\mathcal{K}_{Y,y}^\bullet) = 0 \quad \text{for } \ k \ne j-1.$$

(Indeed, the restriction of any perverse sheaf to a sufficiently small Zariski-open subvariety of its support is a local system shifted by the dimension of the variety.) Moreover, by (1.7.2), we have the isomorphism as vector spaces
$$H^{j-1}(\sigma_{<j}\mathcal{K}_{Y,y}^\bullet) = L_V,$$

and this implies the acyclicity of $\sigma_{\le j}\mathcal{K}_{Y,y}^\bullet$. Since the restriction of the cohomology sheaves $\mathcal{H}^i\sigma_{\le j}\mathcal{K}_Y^\bullet$ to $U^{(j)}$ are locally constant, we see that $\sigma_{\le j}\mathcal{K}_Y^\bullet$ is acyclic on $U^{(j)}$ and hence on $Y^{(1)} \setminus Y^{(j+1)}$ using (A_{j-1}) on $Y^{(1)} \setminus Y^{(j)}$ (since $(j_U)_!\mathbf{Q}_U|_{Y^{(1)}} = 0$). So the shifted mapping cone in (A_j) is supported on $Y^{(j+1)}$, and it remains to show that the shifted mapping cone is a perverse sheaf, i.e. in the abelian category of perverse sheaves (see [1]) we have
$$\mathrm{Coker}((j_U)_!\mathbf{Q}_U[n-1] \to (\sigma_{\le j}\mathcal{K}_Y^\bullet)[n-1]) = 0.$$

But this is clear since the $\mathbf{Q}_{\mathbf{P}(V)}[\dim \mathbf{P}(V)]$ are simple perverse sheaves so that there are no nontrivial subquotients of the perverse sheaf $\mathcal{K}_Y^i[n-1-i]$ supported on $Y^{(j+1)}$ if $i \le j$. Thus we get (A_j), and (1.8.1) follows by induction.

1.9. Remark. Set $r_V = \mathrm{rank}\, L_V$, and $\bar{\mathcal{S}}(D)_V := \{V' \in \bar{\mathcal{S}}(D) \mid V' \supset V\}$. By Lemma (1.8) we have

$$\textstyle\sum_{V'\in\bar{\mathcal{S}}(D)_V} (-1)^{\gamma(V')} r_{V'} = 0. \tag{1.9.1}$$

This implies that $(-1)^{\gamma(V)} r_V$ coincides with the Möbius function defined by increasing induction on $\gamma(V)$, see [24].

Take any D_k, and set
$$\bar{\mathcal{S}}(D)^{(j)}_{\langle k\rangle} = \{V \in \bar{\mathcal{S}}(D)^{(j)} \mid V \not\subset D_k\}, \quad U_k = Y \setminus D_k = \mathbf{C}^{n-1},$$

with the inclusion $j_k : U \to U_k$. Then
$$\mathrm{Gr}^W_{n-1+j}(\mathbf{R}(j_k)_*\mathbf{Q}_U[n-1]) = \textstyle\bigoplus_{V\in\bar{\mathcal{S}}(D)^{(j)}_{\langle k\rangle}} L_V|_{U_k}(-j)[n-1-j].$$

The associated spectral sequence degenerates at E_1, since the $U_k \cap \mathbf{P}(V)$ are affine spaces. This implies that $H^j(U, \mathbf{Q})$ has type (j, j), and we get $F^j H^j(U, \mathbf{C}) = P^j H^j(U, \mathbf{C}) = H^j(U, \mathbf{C})$ for any j, where P is the pole order filtration. This gives examples where $P \neq F$ locally but $P = F$ globally, see [13]. The above assertion is compatible with a result of E. Brieskorn [2] that $H^j(U, \mathbf{Q})$ is generated by logarithmic forms

$$\tfrac{dg_{i_1}}{g_{i_1}} \wedge \cdots \wedge \tfrac{dg_{i_j}}{g_{i_j}},$$

where the g_i are linear functions with constant terms defining $\mathbf{P}(D_i) \setminus \mathbf{P}(D_k) \subset \mathbf{C}^{n-1}$. The E_1-degeneration also implies a formula in [24]

$$b_k(U) = \textstyle\sum_{V \in \bar{\mathcal{S}}(D)^{(j)}_{\langle k\rangle}} r_V. \tag{1.9.2}$$

It is well-known that the Betti numbers $b_k(U)$ are combinatorial invariants of a hyperplane arrangement, see [24]. We have a refinement as follows.

1.10. Proposition. *Set $\mathcal{S} = \mathcal{S}(D)^{\mathrm{nnc}}$, $\mathcal{S}^{D_i} = \{V \in \mathcal{S} \mid V \subset D_i\}$ in the notation of* (1.1). *Then the $b_k(U)$ are determined by $\mathcal{S}, \subset, \mu_{\mathrm{red}}, \gamma$ together with $\mathcal{S}^{D_i}$ $(i \in \Lambda)$.*

Proof. We first show that the r_V are determined by the above combinatorial data. We have $r_V = 1$ for $\bar{\mathcal{S}}(D) \setminus \mathcal{S}$ since $\mathcal{K}^\bullet_{Y,y}$ for $y \notin \mathbf{P}(D^{\mathrm{nnc}})$ is the standard Koszul complex. Since the r_V for $V \in \mathcal{S}$ is determined by induction on $\gamma(V)$ using (1.9.1), it is enough to express

$$\big|\bar{\mathcal{S}}(D)^{(j)}_V \setminus \mathcal{S}\big|,$$

using only the combinatorial data as above. Set $I(V) = \{i \in \Lambda \mid D_i \supset V\}$. Note that $I(V)$ is determined by the $\mathcal{S}^{D_i}$ $(i \in \Lambda)$ if $V \in \mathcal{S}$. Set

$$\begin{aligned} S(\Lambda)^{(j)} &= \{I \subset \Lambda \mid |I| = j\}, \quad S(\Lambda)_V = \{I \subset \Lambda \mid I \subset I(V)\}, \\ S^{\mathrm{nnc}}(\Lambda) &= \{I \subset \Lambda \mid I = I(V') \text{ for some } V' \in \mathcal{S}\}. \end{aligned}$$

Then we have the identification

$$\bar{\mathcal{S}}(D)^{(j)}_V \setminus \mathcal{S} = S(\Lambda)^{(j)} \cap S(\Lambda)_V \setminus S^{\mathrm{nnc}}(\Lambda),$$

and the assertion follows. Thus the r_V are calculated by induction on $\gamma(V)$ using only the above combinatorial data.

Since the Betti number is calculated by using (1.9.2), it is enough to express

$$\big|\bar{\mathcal{S}}(D)^{(j)}_{\langle k\rangle} \setminus \mathcal{S}\big|,$$

by using only the combinatorial data as above. So the assertion follows since

$$\bar{\mathcal{S}}(D)^{(j)}_{\langle k\rangle} \setminus \mathcal{S} = S(\Lambda)^{(j)} \cap S(\Lambda)_{\langle k\rangle} \setminus S^{\mathrm{nnc}}(\Lambda),$$

where $S(\Lambda)_{\langle k\rangle} = \{I \subset \Lambda \mid k \notin I\}$. This finishes the proof of Proposition (1.10).

2. Canonical embedded resolution

The material in this section is treated in a much more general situation in [8]. For the convenience of the reader we treat it under the assumption that $\mathcal{S}$ is stable by intersection. This hypothesis is satisfied in our case, and simplifies certain arguments very much.

2.1. Construction. Let $\mathcal{S}$ be a finite set of proper vector subspaces of the vector space $X = \mathbf{C}^n$ which is stable by intersection (i.e. $V \cap V' \in \mathcal{S}$ if $V, V' \in \mathcal{S}$) and such that $0 \in \mathcal{S}$. We have a function $\gamma : \mathcal{S} \to \mathbf{N}$ associating the codimension of V. There is a natural order on $\mathcal{S}$ defined by the inclusion relation. We say that $\mathcal{S}$ and $\mathcal{S}'$ are combinatorially equivalent if there is a one-to-one correspondence $\mathcal{S} \to \mathcal{S}'$ as ordered sets in a compatible way with γ. Note that a *nested* subset of $\mathcal{S}$ in the sense of [8] is always linearly ordered by the inclusion relation in our paper since $\mathcal{S}$ is stable by intersection.

Let $Y = \mathbf{P}^{n-1}$. For a vector subspace $V \subset X = \mathbf{C}^n$, its corresponding subspace of Y will be denoted by $\mathbf{P}(V)$. For $\mathcal{S}$ as above, there is a sequence of blowing-ups

$$\rho_i : Y_{i+1} \to Y_i \quad \text{for } 0 \le i < n-2,$$

whose center C_i is the *disjoint* union of the proper transforms of $\mathbf{P}(V)$ for $V \in \mathcal{S}$ with $\dim \mathbf{P}(V) = i$, where $Y_0 = Y$. Note that we cannot restrict to the dense edges as in [30] because this is not adequate for our inductive argument.

Set $\widetilde{Y} = Y_{n-2}$ with $\rho : \widetilde{Y} \to Y$ the composition of the ρ_i. We will sometimes denote $\widetilde{Y}$ by $Y^{\mathcal{S}}$. This applies to $\mathbf{P}(V)^{\mathcal{S}^V}$ where Y and $\mathcal{S}$ are replaced by $\mathbf{P}(V)$ and $\mathcal{S}^V$ respectively. Here we define for $V \in \mathcal{S}$

$$\mathcal{S}^V = \{V' \in \mathcal{S} \mid V' \subsetneqq V\}, \quad \mathcal{S}_V = \{V' \in \mathcal{S} \mid V' \supset V\}.$$

For $V \in \mathcal{S} \setminus \{0\}$ with $\dim \mathbf{P}(V) = i$, let $C_{V,j}$ denote the proper transform of $C_{V,0} := \mathbf{P}(V) \subset Y_0$ in Y_j for $1 \le j \le i$. Let $E_{V,i+1}$ be the exceptional divisor of the blow-up along $C_{V,i}$ which is an irreducible component of C_i and is identified with $\mathbf{P}(V)^{\mathcal{S}^V}$ (which is defined above). Let $E_{V,j}$ be the proper transform of $E_{V,i+1}$ in Y_j for $i+1 < j \le n-2$. Finally, set $E_V = E_{V,n-2}$ if $\dim \mathbf{P}(V) < n-2$, and $E_V = C_{V,n-2}$ if $\dim \mathbf{P}(V) = n-2$. For $V = 0$, let E_0 denote the proper transform $\widetilde{H}$ in $\widetilde{Y}$ of a *general* hyperplane H of $\mathbf{P}^{n-1}$. (It is known that the divisor class group of $\widetilde{Y}$ is generated by E_V for $V \in \mathcal{S}$ with $\dim \mathbf{P}(V) < n-2$.)

2.2. Remarks. (i) For smooth varieties $X \supset Y \supset Z$ in general, the proper transform of Y by the blow-up of X along Z is the blow-up of Y along Z.

(ii) For any linear subspaces L, L' of affine space such that $L \cap L' \ne L, L'$, the proper transforms of L and L' by the blow-up along $L \cap L'$ do not intersect.

2.3. Proposition. *The union of E_V for $V \in \mathcal{S} \setminus \{0\}$ is a divisor with normal crossings on $\widetilde{Y}$, and the intersection of E_{V_k} for $V_k \in \mathcal{S} \setminus \{0\}$ with $1 \le k \le r$ is empty unless $V_1 \subset \cdots \subset V_k$ up to a permutation.*

Proof. The last assertion follows from Remark (2.2)(i) because $\mathcal{S}$ is stable by intersection. For the first assertion we take local coordinates $x_1, \ldots, x_{n-1}$ such that V_k is given by $x_i = 0$

for $i > d_k$ where $d_k = \dim V_k$. Then local coordinates $y_1, \dots, y_{n-1}$ of the blow-up along V_1 are given by $y_i = x_i$ if $i \le d_1$ or $i = i_1$, and by $y_i = x_i/x_{i_1}$ otherwise. Here the exceptional divisor of the blow-up is given by $y_{i_1} = 0$ and i_1 is an integer in $(d_1, d_2]$, because we have to consider the proper transforms of the V_k for $k \ge 2$, which are given by $y_i = 0$ for $i > d_k$ if $i_1 \in (d_1, d_2]$. So the assertion follows by repeating this construction.

2.4. Proposition. *The center C_i of the blow-up ρ_i is the disjoint union of $C_{V,i} = \mathbf{P}(V)^{\mathcal{S}^V}$ for $V \in \mathcal{S}$ with $\dim \mathbf{P}(V) = i$, and $C_{V,j+1}$ $(1 \le j < i)$ is the blow-up of $C_{V,j}$ along the disjoint union of $C_{V',j}$ for $V' \in \mathcal{S}^V$ with $\dim \mathbf{P}(V') = j$.*

Proof. This follows from Remarks (2.2)(i) and (ii) using the calculation in (2.3).

2.5. Proposition. *Let $V, V' \in \mathcal{S}$ such that $\dim \mathbf{P}(V) = i < \dim \mathbf{P}(V') = i'$. Then $C_{V',i'}$ is not contained in $E_{V,i'}$, and hence $E_{V,j}$ inductively coincides with the total transform of $E_{V,i}$ for $j > i$.*

Proof. This follows from the above arguments, because $\mathcal{S}$ is stable by intersections and the blow-ups are done by increasing induction on the dimension of the center. More precisely, we have $C_{V',i'} \cap E_{V,i'} = \emptyset$ in the case $V \not\subset V'$. In the other case, repeating the above construction with Y replaced by $\mathbf{P}(V')$, define $C'_{V,i}$, $E'_{V,i+1}$, and E'_V associated to $\mathbf{P}(V) \subset \mathbf{P}(V')$ as in (2.1) above, i.e. $E'_{V,i+1}$ is the exceptional divisor of the blow-up along the center $C'_{V,i}$, and E'_V is the proper transform of $E'_{V,i+1}$ in $\mathbf{P}(V')^{\mathcal{S}^{V'}} = C_{V',i'}$, where the upper script $'$ of $C'_{V,i}$, $E'_{V,i+1}$ and E'_V means that the construction is done for $\mathbf{P}(V')$ instead of Y. Then $C_{V',i'} \cap E_{V,i'} = E'_V$. This finishes the proof of Proposition (2.5).

2.6. Proposition. *For $V \in \mathcal{S}$, E_V depends only on $\mathcal{S}^V$ and $\mathcal{S}_V$, and there is a canonical decomposition*

$$E_V = \mathbf{P}(V)^{\mathcal{S}^V} \times \mathbf{P}(X/V)^{\mathcal{S}_V},$$

where $\mathbf{P}(V)^{\mathcal{S}^V}$, $\mathbf{P}(X/V)^{\mathcal{S}_V}$ are the successive blow-ups of $\mathbf{P}(V)$ and $\mathbf{P}(X/V)$ respectively associated with $\mathcal{S}^V$ and $\mathcal{S}_V$ as in (2.1).

Proof. The first assertion is clear because $\mathcal{S}$ is stable by intersections. Let r be the codimension of $\mathbf{P}(V)$ in Y (i.e. $r = n-1-i$). Taking r general hyperplanes containing V, and considering their proper transforms whose intersection is $C_{V,i}$, we see that the tensor of the conormal bundle of $C_{V,i}$ with some line bundle is a trivial vector bundle. Hence $E_{V,i+1}$ is a trivial $\mathbf{P}^{r-1}$-bundle over $C_{V,i}$ with a canonical projection to $\mathbf{P}^{r-1}$ (which is independent of the choice of the hyperplanes up to the action of $PGL(r-1, \mathbf{C})$). So the assertion follows, because $\mathcal{S}_V$ is identified with a set of vector subspaces of X/V.

2.7. Proposition. *Let $V_k \in \mathcal{S}$ for $1 \le k \le r$ such that $0 \ne V_1 \subsetneqq \cdots \subsetneqq V_r$. Then*

$$\textstyle\bigcap_{1 \le k \le r} E_{V_k} = \prod_{0 \le k \le r} \mathbf{P}(V_{k+1}/V_k)^{\mathcal{S}_k},$$

where $V_0 = 0, V_{r+1} = X$, and $\mathcal{S}_k = \{V/V_k \mid V \in \mathcal{S}$ with $V_k \subset V \subsetneqq V_{k+1}\}$.

Proof. Let $V = V_r$ and $i = \dim \mathbf{P}(V)$. Then $E_V \cap E_{V_k}$ is the pull-back of $C_{V,i} \cap E_{V_k,i}$ by the projection $E_V \to C_{V,i}$ for $1 \le k \le r-1$. So the assertion follows from (2.6)

by induction on r, where the inductive hypothesis is applied to $C_{V,i} = \mathbf{P}(V)^{\mathcal{S}^V}$ and $C_{V,i} \cap E_{V_k,i}\,(1 \le k \le r-1)$ which are calculated in the proof of Proposition (2.5).

2.8. Proposition. *For $V' \in \mathcal{S}^V$ (resp. $\mathcal{S}_V$) such that $V' \ne V$, the pull-back of $E_{V'}$ to E_V coincides with the pull-back of $E'_{V'}$ on $\mathbf{P}(V)^{\mathcal{S}^V}$ by pr_1 (resp. of $E''_{V'}$ on $\mathbf{P}(X/V)^{\mathcal{S}_V}$ by pr_2). Here $\mathbf{P}(V)^{\mathcal{S}^V}$ and $\mathbf{P}(X/V)^{\mathcal{S}_V}$ are as in (2.6), and pr_i denote the projection to the i-th factor. For $V' = V$, the pull-back of E_V to E_V as a divisor class is given by*

$$pr_1^*(\widetilde{H}' - \textstyle\sum_{V' \in \mathcal{S}^V} E'_{V'}) - pr_2^* \widetilde{H}'',$$

where $\widetilde{H}'$ (resp. $\widetilde{H}''$) is the proper transform of a general hyperplane of $\mathbf{P}(V)$ (resp. $\mathbf{P}(X/V)$), and it is zero if $\dim \mathbf{P}(V) = 0$ (resp. $\dim \mathbf{P}(X/V) = 0$).

Proof. The first assertion follows from (2.5–6). For the second, take a general hyperplane of Y intersecting $\mathbf{P}(V)$ transversally and also a hyperplane of Y containing $\mathbf{P}(V)$ and corresponding to a general hyperplane of $\mathbf{P}(X/V)$. Then the assertion follows by considering the difference between their pull-backs to $\widetilde{Y}$.

3. Proofs of Theorems 3 and 5

3.1. Proof of Theorem 3. This follows from Proposition (1.5) together with the Riemann-Roch theorem for surfaces (see e.g. [17], Example 15.2.2)

$$\chi(\mathcal{O}_{\widetilde{Y}}(D_k)) = \tfrac{1}{2}\, D_k \cdot (D_k - K_{\widetilde{Y}}) + \chi(\mathcal{O}_{\widetilde{Y}}). \tag{3.1.1}$$

Here $\widetilde{Y}$ is the blow-up of $Y = \mathbf{P}^2$ along the points of $\mathbf{P}(D^{\mathrm{nnc}})$ in the notation of (0.1), and D_k is a divisor on it (which will be defined later).

By (1.4.1) we may assume

$$\alpha = \frac{i}{d} + \ell \quad \text{with } i = d - k \in [1, d],\ \ell = 2 - p \in [0, 2],$$

since $n_{f,\alpha} = 0$ for the other α. Consider first the case $\ell = 0$. We have

$$\Omega^2_{\widetilde{Y}}(\log \widetilde{Z}) = \mathcal{O}_{\widetilde{Y}}((d-3)\widetilde{H} + \textstyle\sum_{j \in J''} (2 - m_j) E_j)),$$

since $\Omega^2_{\widetilde{Y}} = \rho^* \Omega^2_Y \otimes_{\mathcal{O}} \mathcal{O}_{\widetilde{Y}}(\sum_{j \in J''} E_j)$ and $\widetilde{Z}_{\mathrm{red}} = \rho^* Z + \sum_{j \in J''} (1 - m_j) E_j$ in the notation of (1.4). So we apply the Riemann-Roch theorem (3.1.1) to

$$D_k = (d - k - 3)\widetilde{H} + \textstyle\sum_{j \in J''} (-m_j + \lfloor k m_j / d \rfloor + 2) E_j,$$

using Proposition (1.5) for $p = 2$. We have $D_k = \sum_j A_j E_j + C \widetilde{H}$ with

$$A_j = 2 + \lfloor -i m_j / d \rfloor = 2 - \lceil i m_j / d \rceil, \quad C = i - 3,$$

and $K_{\widetilde{Y}} = -3\widetilde{H} + \sum_j E_j$. So we get

$$D_k^2 = -\sum_j A_j^2 + C^2, \quad D_k \cdot K_{\widetilde{Y}} = -\sum_j A_j - 3C,$$

since $\widetilde{H}^2 = 1$, $E_j^2 = -1$, and $\widetilde{H}, E_j$ are orthogonal to each other. These imply the first equality by (3.1.1) since $\chi(\mathcal{O}_{\widetilde{Y}}) = 1$.

The argument is similar for the last equality where $\ell = 2$, $p = 0$, $D_k = \sum_j A_j E_j + C\widetilde{H}$ with

$$A_j = \lfloor km_j/d \rfloor = m_j - \lceil im_j/d \rceil, \quad C = -k = i - d.$$

Note that the reduced cohomology is used for the definition of spectrum, and the difference corresponds to $\delta_{i,d}$ in the case $i = d$.

By the identity $\binom{a+b}{2} - \binom{a}{2} - \binom{b}{2} = ab$, the middle equality for $\ell = 1$ follows from the others since we have by (1.4.2)

$$\sum_{\ell=0}^{2} n_{f,\frac{i}{d}+\ell} = \chi(U) - \delta_{i,d} \quad \text{with} \quad \chi(U) = \binom{d-2}{2} - \sum_{m\geq 3} \nu_m^{(2)} \binom{m-1}{2}.$$

Here the first equality is clear by the definition of spectrum. The last equality is shown by using a small deformation to a generic central arrangement D' where $\mathbf{P}(D')$ is a divisor with normal crossing so that

$$\chi(\mathbf{P}^2 \setminus \mathbf{P}(D')) = 3 - 2d - \binom{d}{2} = \binom{d-2}{2}.$$

The difference of the local Euler characteristics of $\mathbf{P}(D)$ and $\mathbf{P}(D')$ at each point of $\mathbf{P}(D)$ with multiplicity m is given by $1 - \left(m - \binom{m}{2}\right) = \binom{m-1}{2}$. So the assertion follows.

3.2. Generic case. Assume D is a generic central hyperplane arrangement, i.e. $\mathbf{P}(D) \subset \mathbf{P}^{n-1}$ is a divisor with normal crossings. In this case it is known ([27], Cor. 1) that

$$n_{f,\alpha} = n_{f,n-\alpha} = \binom{i-1}{n-1} \quad \text{for} \quad \alpha = i/d < 1,$$

where $n_{f,\alpha} = 0$ for $d\alpha \notin \mathbf{Z}$. For $\alpha = i \in \mathbf{Z}$, we have by 5.6.1 in loc. cit.

$$n_{f,i} = (-1)^{i-1} \binom{d-1}{n-i} \quad \text{for} \quad 1 \leq i \leq n-1.$$

It is possible to calculate $n_{f,\alpha}$ for any α using Proposition (1.5) and the Bott vanishing theorem.

3.3. Mustaţă's formula. In the notation of (1.1), set $\mathcal{S}' = \mathcal{S}(D^{\mathrm{nrnc}})$. For each $V \in \mathcal{S}'$, let $\mathcal{I}_V \subset \mathbf{C}[X]$ be the reduced ideal of V. Mustaţă's formula [22] states that for any $\alpha > 0$

$$\mathcal{J}(X, \alpha D) = \bigcap_{V\in\mathcal{S}'} \mathcal{I}_V^{\lfloor \alpha\mu(V) \rfloor - \gamma(V)+1}. \tag{3.3.1}$$

In the nonreduced case this is due to Z. Teitler [32] (see also [27], 2.2).

If $V = 0$, then we have for $\alpha = j/d$ with $j \in [1, d] \cap \mathbf{Z}$

$$\mathcal{I}_0^{\lfloor (\alpha-\varepsilon)\mu(0) \rfloor - (\gamma(0))+1} = \mathcal{I}_0^{j-n},$$

for $0<\varepsilon\ll 1/d$, since $\gamma(0)=n$ and $\mu(0)=\deg D=d$. So we get (0.1) where $n=3$, see also [22], Cor. 2.1.

3.4. Proof of Theorem 5. By (1.4.3–5) for $p=n-1$, we get for $\alpha=\frac{i}{d}\in(0,1]$ with $i=d-k\in[1,d]$

$$\begin{aligned} n_{f,\alpha}&=\dim\Gamma\big(\widetilde{Y},\mathcal{O}_{\widetilde{Y}}((i-n)\widetilde{H}+\textstyle\sum_{j\in J}(c_j-\lceil im_j/d\rceil)E_j)\big)\\ &=\dim\big(\textstyle\bigcap_{V\in\mathcal{S}'\setminus\{0\}}\mathcal{I}_V^{\lceil\alpha\mu(V)\rceil-\gamma(V)}\cap\mathbf{C}[X]_{i-n}\big),\end{aligned}\tag{3.4.1}$$

where the second equality is shown by using the injection

$$\begin{aligned}&\Gamma\big(\widetilde{Y},\mathcal{O}_{\widetilde{Y}}((i-n)\widetilde{H}+\textstyle\sum_j\big(c_j-\lceil im_j/d\rceil\big)E_j\big)\\ &\quad\subset\Gamma(\widetilde{Y},\mathcal{O}_{\widetilde{Y}}((i-n)\widetilde{H}))=\Gamma(Y,\mathcal{O}_Y(i-n))=\mathbf{C}[X]_{i-n}.\end{aligned}$$

Indeed, for $g\in\mathbf{C}[X]_{i-n}=\Gamma(\widetilde{Y},\mathcal{O}_{\widetilde{Y}}((i-n)\widetilde{H}))$, the condition $g\in\mathcal{I}_V^k$ corresponds to that $\pi^*g\in\mathcal{I}_{E_j}^k$ if E_j corresponds to V, where $\mathcal{I}_{E_j}$ is the ideal of E_j and

$$k=\lceil im_j/d\rceil-c_j=\lceil\alpha\mu(V)\rceil-\gamma(V).$$

Note that we may have $c_j-\lceil im_j/d\rceil>0$ only in the case $c_j\ge 2$ so that g cannot have a pole. Thus the assertion follows.

3.5. Relation with b-functions. It does not seem easy to get an explicit formula for the jumping coefficients and the spectrum of a hyperplane arrangement in the general case. However, it seems to be more difficult to calculate the roots of the b-function of a hyperplane arrangement except for the case of a generic central arrangement [33], see also [27]. The relation between the jumping coefficients $\mathrm{JC}(D)$ and the roots of the b-function R_f is quite complicated although there is an inclusion relation

$$\mathrm{JC}(D)\cap(0,1)\subset R_f\cap(0,1),$$

as is shown in [14]. The converse inclusion holds under some conditions, see [27]. However, it does not hold without these conditions as is shown by the following.

3.6. Example. Let $n=3$, $d=7$ and $f=(x^2-y^2)(x^2-z^2)(y^2-z^2)z$. By (0.1) or Theorem 3 we have $\frac{5}{7}\notin\mathrm{JC}(D)$, but $\frac{5}{7}\in R_f$, see [28]. In this case

$$\dim\mathbf{C}[x,y,z]_2=\#\mathbf{P}(D^{\mathrm{nnc}})=6.\tag{3.6.1}$$

So we have to prove the non-degeneracy of some matrix to show the non-existence of a nontrivial homogeneous polynomial of degree 2 vanishing at all the 6 points if we want to show that $\frac{5}{7}\notin\mathrm{JC}(D)$ using (0.1). Note that $n_{f,5/7}=0$ by Theorem 3 where $d=7$, $\nu_3^{(2)}=6$ and $\nu_m^{(2)}=0\,(m>3)$. This implies that $\frac{5}{7}\notin\mathrm{JC}(D)$ by Proposition (4.2).

4. Proofs of Theorems 1–4 by induction

4.1. Isolated jumping coefficients. Let X be a smooth variety (or a complex manifold), and D be a divisor on it. Let $\mathcal{J}(X,\alpha D)$ denote the multiplier ideals for $\alpha>0$, see [21]. The graded pieces of the multiplier ideals are defined for $\alpha>0$ with $0<\varepsilon\ll 1$ by

$$\mathcal{G}(X,\alpha D)=\mathcal{J}(X,(\alpha-\varepsilon)D)/\mathcal{J}(X,\alpha D).$$

The jumping coefficients are the rational numbers α such that $\mathcal{G}(X,\alpha D)\neq 0$. We say that α is an *isolated* jumping coefficient at x if $\mathcal{G}(X,\alpha D)$ is supported on x.

If D has an isolated singularity at x and is defined locally by f, then the coefficient $n_{f,\alpha}$ of the spectrum $\mathrm{Sp}(f)=\sum_{\alpha>0}n_{f,\alpha}t^{\alpha}$ for $\alpha\in(0,1)$ is given (see [3]) by

$$n_{f,\alpha}=\dim\mathcal{G}(X,\alpha D)_x. \tag{4.1.1}$$

4.2. Proposition. *The assertion* (4.1.1) *holds by assuming only that* $\alpha\in(0,1)$ *is an isolated jumping coefficient at* x.

Proof. By [6], we have a canonical isomorphism

$$\mathcal{G}(X,\alpha D)=F_{-n}\mathrm{Gr}_V^{\alpha}\mathcal{B}_f,$$

where $\mathrm{Gr}_V^{\alpha}\mathcal{B}_f$ coincides with the λ-eigenspace of $\psi_f\mathcal{O}_X$ by the action of the monodromy where $\lambda=e^{-2\pi i\alpha}$. Here we have to show that F_{-n} does not change by taking the pull-back by $i_x:\{x\}\to X$ as in (1.6). Choosing local coordinates $x_1,\dots,x_n$ of (X,x) and using [26], 2.24, the pull-back i_x^* is given by iterating

$$i_k^*=C(\mathrm{can}:\psi_{x_k,1}\to\varphi_{x_k,1}).$$

For the underlying filtered left D-modules (M,F), the last functor is given by the mapping cone of

$$\partial_{x_k}:\mathrm{Gr}_{V_k}^1(M,F[1])\to\mathrm{Gr}_{V_k}^0(M,F),$$

where V_k is the V-filtration along $x_k=0$ and $x_k\partial_k-\alpha$ is nilpotent on $\mathrm{Gr}_{V_k}^{\alpha}$. Since $\mathrm{supp}\,F_{-n}M=\{x\}$, we see that $F_{-n}M$ is contained in

$$(i_x)_*H^0i_x^!M=\Gamma_{[x]}M\subset M,$$

which underlies a mixed Hodge module, and is isomorphic to $\bigoplus_i(\mathbf{C}[\partial_1,\dots,\partial_n],F[p_i])$ with $p_i\in\mathbf{Z}$. (Indeed, it is the direct image as a D-module of the filtered vector space $H^0i_x^!(M,F)$ by the closed embedding i_x. Note that $\mathbf{C}[\partial_1,\dots,\partial_n]$ has the Hodge filtration F by the degree of polynomials in ∂_i.) So $F_{-n}M$ does not change by passing to $\mathrm{Gr}_{V_k}^0(M,F)$ inductively. Since $F_{-n}=0$ on $(M,F[1])$, this implies the desired result.

4.3. Proof of Proposition 1. Assume $\alpha\in\mathrm{JC}(D)\cap(0,1)$. It is well-known that $\mathcal{G}(X,\alpha D)=0$ for $\alpha\in(0,1)$ if D is a reduced divisor with normal crossings. So the

support of $\mathcal{G}(X, \alpha D)$ is a union of $V \in \mathcal{S}(D)^{\rm nrnc}$, since D is locally trivial along a non-empty Zariski open subset of V. Restricting D to an affine subspace which is transversal to the Zariski open subset of $V \in \mathcal{S}(D)^{\rm nrnc}$ and has complementary dimension with V, it is enough to consider the case of isolated jumping coefficients, since D is locally the product of its restriction to the transversal space with V. (But this does not mean that the singular points of D are 0-dimensional.) Then we get the non-vanishing of $n_{f_{X/V},\alpha}$ by Proposition (4.2) since $f_{X/V}$ is identified with the defining polynomial of the restriction of D to the transversal space. So Proposition 1 follows since the opposite implication is well-known, see [3].

4.4. Theorem. *In the notation of* (1.1) *and* (2.1), *set* $\mathcal{S} = \mathcal{S}(D)^{\rm nnc}$ *and* $\mathcal{S}^{D_i} = \{V \in \mathcal{S} \mid V \subset D_i\}$. *Let* $\widetilde{Z}$ *be as in* (1.4). *For* $a = (a_V)_{V\in\mathcal{S}} \in \mathbf{Z}^{\mathcal{S}}$ *define*

$$\Phi^p_{\mathcal{S}}(a) = \chi(\widetilde{Y}, \Omega^p_{\widetilde{Y}}(\log \widetilde{Z}) \otimes_{\mathcal{O}} \mathcal{O}_{\widetilde{Y}}(\textstyle\sum_{V\in\mathcal{S}} a_V E_V)).$$

Then $\Phi^p_{\mathcal{S}}(a)$ *is a polynomial in* $a = (a_V)_{V\in\mathcal{S}}$ *whose coefficients are rational numbers and are determined by the combinatorial data of* D. *More precisely, it depends only on* $\mathcal{S}, \subset, \mu_{\rm red}, \gamma$ *together with* $\mathcal{S}^{D_i}$ $(i \in \Lambda)$ *in the notation of* (1.1.1). *If* $p = 0$, *then* $\Phi^0_{\mathcal{S}}(a)$ *depends only on the weak combinatorial data.*

Proof. We show this by increasing induction on $n = \dim X \geq 2$. First we show the assertion on $\Phi^p_{\mathcal{S}}(a)$ for any p. If $n = 2$, the assertion is trivial by the Riemann-Roch theorem for curves. Here $\mathcal{S} = \{0\}$ and the number of the points of $\widetilde{Z}$ is enough for the calculation.

Assume $n > 2$, and set

$$M(a) = \Omega^p_{\widetilde{Y}}(\log \widetilde{Z}) \otimes_{\mathcal{O}} \mathcal{O}_{\widetilde{Y}}(\textstyle\sum_{V\in\mathcal{S}} a_V E_V).$$

Here we may assume $p < \dim \widetilde{Y}$ since the case $p = \dim \widetilde{Y}$ is reduced to the case $p = 0$. Take some $V \in \mathcal{S}$, and set $E = E_V$ if $V \neq 0$. In the case $V = 0$, set $E = \widetilde{H}$ which is the pull-back of a sufficiently general hyperplane. We have a short exact sequence

$$0 \to M(a - \mathbf{1}_V) \to M(a) \to M(a) \otimes_{\mathcal{O}_{\widetilde{Y}}} \mathcal{O}_E \to 0,$$

where $\mathbf{1}_V \in \mathbf{Z}^{\mathcal{S}}$ is defined by $(\mathbf{1}_V)_{V'} = 0$ for $V' \neq V$ and $(\mathbf{1}_V)_V = 1$. So we get

$$\Phi^p_{\mathcal{S}}(a) - \Phi^p_{\mathcal{S}}(a - \mathbf{1}_V) = \chi(E, M(a) \otimes_{\mathcal{O}_{\widetilde{Y}}} \mathcal{O}_E). \tag{4.4.1}$$

Using the identity $\binom{x}{k} - \binom{x-1}{k} = \binom{x-1}{k-1}$ as polynomials in x where $k \in \mathbf{Z}_{>0}$ (see also [18], I, Prop. 7.3(a)), it is enough to show that (4.4.1) is a polynomial determined by the combinatorial data.

We consider first the case $V \neq 0$. Let N^*_E denote the conormal bundle of $E \subset \widetilde{Y}$. This is the restriction of the line bundle $\mathcal{O}_{\widetilde{Y}}(-E)$, and the restriction as a divisor is calculated

in (2.8). Let $\widetilde{Z}'$ be the closure of $\widetilde{Z}\setminus E$. Set $\widetilde{Z}'_E=\widetilde{Z}'\cap E$. There is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} & & 0 & \to & \mathcal{O}_{\widetilde{Y}}(-E) & = & \mathcal{O}_{\widetilde{Y}}(-E) & \to & 0 \\ & & \downarrow & & \cap & & \cap & & \\ 0 & \to & \Omega^1_{\widetilde{Y}}(\log\widetilde{Z}') & \to & M & \to & \mathcal{O}_{\widetilde{Y}} & \to & 0 \\ & & || & & \downarrow & & \downarrow & & \\ 0 & \to & \Omega^1_{\widetilde{Y}}(\log\widetilde{Z}') & \to & \Omega^1_{\widetilde{Y}}(\log\widetilde{Z}) & \to & \mathcal{O}_E & \to & 0 \end{array}$$

where $M:=\mathrm{Ker}(\Omega^1_{\widetilde{Y}}(\log\widetilde{Z})\oplus\mathcal{O}_{\widetilde{Y}}\to\mathcal{O}_E)$. Taking the pull-back by $E\to\widetilde{Y}$, we get short exact sequences

$$0\to N_E^*\to M|_E\to\Omega^1_{\widetilde{Y}}(\log\widetilde{Z})|_E\to 0,$$

$$0\to\Omega^1_{\widetilde{Y}}(\log\widetilde{Z}')|_E\to M|_E\to\mathcal{O}_E\to 0.$$

We have also a short exact sequence

$$0\to N_E^*\to\Omega^1_{\widetilde{Y}}(\log\widetilde{Z}')|_E\to\Omega^1_E(\log(\widetilde{Z}'_E)\to 0. \tag{4.4.2}$$

These imply the equalities in the Grothendieck group

$$\begin{aligned}\big[\textstyle\bigwedge^p M|_E\big]&=\big[\Omega^p_{\widetilde{Y}}(\log\widetilde{Z})|_E\big]+\big[N_E^*\otimes\Omega^{p-1}_{\widetilde{Y}}(\log\widetilde{Z})|_E\big],\\ \big[\textstyle\bigwedge^p M|_E\big]&=\big[\Omega^p_{\widetilde{Y}}(\log\widetilde{Z}')|_E\big]+\big[\Omega^{p-1}_{\widetilde{Y}}(\log\widetilde{Z}')|_E\big]\\ &=\big[\Omega^p_E(\log\widetilde{Z}'_E)\big]+\big[N_E^*\otimes\Omega^{p-1}_E(\log\widetilde{Z}'_E)\big]\\ &\quad+\big[\Omega^{p-1}_E(\log\widetilde{Z}'_E)\big]+\big[N_E^*\otimes\Omega^{p-2}_E(\log\widetilde{Z}'_E)\big].\end{aligned}$$

So we get by increasing induction on p

$$\big[\Omega^p_{\widetilde{Y}}(\log\widetilde{Z})|_E\big]=\big[\Omega^p_E(\log\widetilde{Z}'_E)\big]+\big[\Omega^{p-1}_E(\log\widetilde{Z}'_E)\big]. \tag{4.4.3}$$

The assertion on (4.4.1) is thus reduced to that

$$\chi(E,\Omega^p_E(\log\widetilde{Z}'_E)\otimes_{\mathcal{O}}\mathcal{O}_{\widetilde{Y}}(\textstyle\sum_{V\in\mathcal{S}}a_VE_V)|_E)$$

is a polynomial determined by the combinatorial data. By Proposition (2.6) we have

$$E=\mathbf{P}(V)^{\mathcal{S}^V}\times\mathbf{P}(X/V)^{\mathcal{S}_V},$$

and the restriction of $\mathcal{O}_{\widetilde{Y}}(\sum_{V\in\mathcal{S}}a_VE_V))$ to E is calculated by Proposition (2.8). Moreover we have the decomposition

$$\widetilde{Z}'_E=pr_1^*Z_1+pr_2^*Z_2,$$

where Z_1 is given by $E'_{V'}$ with $V'\subset V$, and Z_2 is given by $E''_{D_i/V},E''_{V'/V}$ with $D_i,V'\supset V$. Here $E''_{D_i/V}\subset\mathbf{P}(X/V)^{\mathcal{S}_V}$ is the proper transform of $\mathbf{P}(D_i/V)\subset\mathbf{P}(X/V)$, and the

associated combinatorial data $(\mathcal{S}_V)^{D_i/V}$ is given by $\mathcal{S}_V \cap \mathcal{S}^{D_i}$. So the assertion after (4.4.1) for $V \ne 0$ follows from the inductive assumption using the Künneth-type decomposition of $\Omega_E^p(\log \widetilde{Z}'_E)$.

For $V = 0$ we have a similar assertion since $\widetilde{H}$ intersects transversally E_V for every $V \in \mathcal{S} \setminus \{0\}$. Using an exact sequence similar to (4.4.2), we get instead of (4.4.3)

$$\big[\Omega_{\widetilde{Y}}^p(\log \widetilde{Z})|_{\widetilde{H}}\big] = \big[\Omega_{\widetilde{H}}^p(\log \widetilde{Z}_{\widetilde{H}})\big] + \big[N_{\widetilde{H}}^* \otimes \Omega_{\widetilde{H}}^{p-1}(\log \widetilde{Z}_{\widetilde{H}})\big]. \tag{4.4.4}$$

Applying the inductive hypothesis, we can then calculate

$$\chi(\widetilde{H}, \Omega_{\widetilde{Y}}^p(\log \widetilde{Z})|_{\widetilde{H}} \otimes_{\mathcal{O}} \mathcal{O}_{\widetilde{Y}}(\textstyle\sum_{V \in \mathcal{S}} a_V E_V)|_{\widetilde{H}}),$$

where $\mathcal{S}^{D_i}$ and $\mathcal{S}$ are replaced by those obtained by deleting the 1-dimensional V (i.e. $\dim \mathbf{P}(V) = 0$). So the assertion after (4.4.1) for $V = 0$ is also proved.

Thus the assertion is reduced to the case $a = 0$, and we have to show that

$$\Phi_{\mathcal{S}}^p(0) = \chi(\widetilde{Y}, \Omega_{\widetilde{Y}}^p(\log \widetilde{Z}))$$

depends only on the combinatorial data. It is known that each $H^j(U)$ is generated by products of logarithmic 1-forms (see [2]), and hence has type (j,j). Then the assertion that the Hodge numbers of $U = \widetilde{Y} \setminus \widetilde{Z}$ depend only on the combinatorial data is equivalent to a similar assertion for the Betti numbers. So the assertion follows from Proposition (1.10).

The argument is similar and easier for the assertion on the weak combinatorial data in the case $p = 0$, since we do not have to treat the logarithmic forms by the isomorphism $\Omega_{\widetilde{Y}}^0(\log \widetilde{Z}) = \mathcal{O}_{\widetilde{Y}}$. This finishes the proof of Theorem (4.4).

4.5. Proofs of Theorems 1–2 by induction. Theorem 1 follows from Theorem (4.4) and Proposition (1.5). If D is reduced, then $\lfloor km_j/d \rfloor = 0$ for $j \in J'$ in the notation of (1.4), and $c_j - m_j = 0$ for $j \in J'$ in (1.4.5). Using the second equality of (1.4.5) for $p = n-1$, Theorem 2 then follows from Theorem (4.4) and Proposition (1.5).

4.6. Remark. In the original version [29], the argument in the proof of Theorem (4.4) treated $\Omega_{\widetilde{Y}}^p$ and not $\Omega_{\widetilde{Y}}^p(\log \widetilde{Z})$. By this method we have to take the graded pieces of the weight filtration W on the logarithmic forms, and the argument becomes more complicated. The above proof of Theorem 1 was obtained after the new proof in the next section appeared.

We can calculate examples using the method in this section as is shown below.

4.7. Proof of Theorem 3 by induction. Let $\Phi^p(A, C)$ denote $\Phi_{\mathcal{S}}^p(a)$ in (4.4) where $A = (A_j)$, and the a_V are denoted by A_j or C depending on whether $\dim V = 1$ or 0. Let E_j denote the exceptional divisor corresponding to A_j. Then

$$\Phi(A, C) = \chi(\widetilde{Y}, \mathcal{E}) \quad \text{with} \quad \mathcal{E} = \mathcal{O}_{\widetilde{Y}}(\textstyle\sum_j A_j E_j + C\widetilde{H}).$$

We have $\mathbf{R}\pi_* \mathcal{O}_{\widetilde{Y}} = \mathcal{O}_Y$ where $\pi : \widetilde{Y} \to Y$ since Y is nonsingular. So we first get

$$\Phi^0(0, C) = \tbinom{C+2}{2}.$$

Fix j, and let $\mathbf{1}_j$ be as in the proof of Theorem (4.4). We have $\mathcal{O}_{\widetilde{Y}}(E_j)|_{E_j} = \mathcal{O}_{E_j}(-1)$ where $E_j = \mathbf{P}^1$. (This is shown by using the total transform of a hyperplane passing through the center of the blow-up.) Hence

$$\Phi^0(A, C) - \Phi^0(A - \mathbf{1}_j, C) = \chi(E_j, \mathcal{O}_{E_j}(-A_j)) = 1 - A_j.$$

Thus we get

$$\Phi^0(A, C) = \tbinom{C+2}{2} - \textstyle\sum_j \tbinom{A_j}{2}.$$

This implies the assertions for $\alpha \in (0, 1] \cup (2, 3]$ by setting A_j, C as in (3.1).

For $p = 1$, we have by (4.4.4) applied to a general $H = \mathbf{P}^1 \subset Y$

$$\begin{aligned}\Phi^1(0, C) - \Phi^1(0, C - 1) &= \chi(\mathbf{P}^1, \Omega^1_{\mathbf{P}^1} \otimes \mathcal{O}_{\mathbf{P}^1}(C + d)) + \chi(\mathbf{P}^1, N^*_H \otimes \mathcal{O}_{\mathbf{P}^1}(C))\\ &= (C + d - 1) + C = 2C + d - 1.\end{aligned}$$

Since $\Phi^1(0, 0) = b_1(U) = d - 1$, we get

$$\Phi^1(0, C) = C^2 + dC + d - 1.$$

Fix now j. We have by (4.4.3)

$$\begin{aligned}\Phi^1(A, C) - \Phi^1(A - \mathbf{1}_j, C) &= \chi(\mathbf{P}^1, \Omega^1_{\mathbf{P}^1} \otimes \mathcal{O}_{\mathbf{P}^1}(m_j - A_j)) + \chi(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-A_j))\\ &= (m_j - A_j - 1) + (1 - A_j) = m_j - 2A_j.\end{aligned}$$

We get thus

$$\Phi^1(A, C) = \textstyle\sum_j (-A_j^2 - A_j + m_j A_j) + C^2 + dC + d - 1.$$

Here $A_j = m_j - \lceil i m_j / d \rceil$, $C = i - d$ by Proposition (1.5). So the assertion follows.

4.8. Proof of Theorem 4 by induction. Let $\Phi(A, B, C)$ denote $\Phi^0_{\mathcal{S}}(a)$ in (4.4) where $A = (A_j)$, $B = (B_k)$, and A_j, B_k, C denote a_V depending on whether $\dim V = 2, 1, 0$. Let a_j, b_k, c denote the corresponding divisor classes so that $\Phi(A, B, C) = \chi(\widetilde{Y}, \mathcal{E})$ with

$$\mathcal{E} = \mathcal{O}_{\widetilde{Y}}(\textstyle\sum_j A_j a_j + \sum_k B_k b_k + Cc).$$

We have $\mathbf{R}\pi_* \mathcal{O}_{\widetilde{Y}} = \mathcal{O}_Y$ where $\pi : \widetilde{Y} \to Y$ since Y is nonsingular. So we first get

$$\Phi(0, 0, C) = \tbinom{C+3}{3}.$$

Applying the short exact sequence to an exceptional divisor which is isomorphic to $\widetilde{\mathbf{P}}^2$ and corresponds to some b_k, we get then inductively

$$\Phi(0, B, C) = \textstyle\sum_k \tbinom{B_k}{3} + \tbinom{C+3}{3},$$

using $\binom{B_k}{3} - \binom{B_k - 1}{3} = \binom{B_k - 1}{2} = \binom{2 - B_k}{2}$. Indeed, the restriction of the exceptional divisor to itself is the hyperplane section class up to a sign using a general hyperplane of $\mathbf{P}^3$

passing through the point corresponding to b_k. Then we can use the same argument as above.

We apply the same argument to an exceptional divisor which is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ and corresponds to some a_j. Let n_j be the number of b_k with $k \subset j$. Here we write $k \subset j$ when there is an inclusion between the corresponding V. Let $\mathbf{1}_j$ be as in the proof of Theorem (4.4). We have to calculate

$$\Phi(A, B, C) - \Phi(A - \mathbf{1}_j, B, C) = \chi(E, \mathcal{E}|_E).$$

Let e_1, e_2 respectively denote the class of $pt \times \mathbf{P}^1$ and $\mathbf{P}^1 \times pt$. Since the restrictions of a_j, b_k $(k \subset j)$ and c are respectively

$$(1 - n_j)e_1 - e_2, \quad e_1, \quad e_1,$$

the restriction of $\sum_j A_j a_j + \sum_k B_k b_k + Cc$ to $\mathbf{P}^1 \times \mathbf{P}^1$ is

$$\big((1 - n_j)A_j + \textstyle\sum_{k \subset j} B_k + C\big)e_1 - A_j e_2.$$

Then

$$\begin{aligned} \chi(E, \mathcal{E}|_E) &= \big((1 - n_j)A_j + \textstyle\sum_{k \subset j} B_k + C + 1\big)(1 - A_j) \\ &= 2(n_j - 1)\binom{A_j}{2} - (A_j - 1)\big(\textstyle\sum_{k \subset j} B_k + C + 1\big). \end{aligned}$$

So we get

$$\chi(\mathcal{E}) = \textstyle\sum_j \Big(2(n_j - 1)\binom{A_j+1}{3} - \binom{A_j}{2}\big(\textstyle\sum_{k \subset j} B_k + C + 1\big)\Big) + \textstyle\sum_k \binom{B_k}{3} + \binom{C+3}{3}. \tag{4.8.1}$$

We apply this to $\mathcal{E}$ with $A_j = 2 - \lceil im_j/d \rceil$, $B_j = 3 - \lceil im_j/d \rceil$, $C = i - 4$, where $i = d - k$ and $p = 3$ in Proposition (1.5), see (1.4.5). Then Theorem 4 follows.

5. Proofs of Theorems 1–4 by HRR

5.1. Hirzebruch-Riemann-Roch Theorem. For a vector bundle $\mathcal{E}$ of rank r on a smooth complex projective variety X, there are Chern classes $c_i(\mathcal{E}) \in H^{2i}(X, \mathbf{Q})$ such that $c_0(\mathcal{E}) = 1$, $c_i(\mathcal{E}) = 0$ for $i > r$, and the following facts are well known (see [17], [19]):

(a) The total Chern class, the Chern character, and the Todd class are defined by

$$c(\mathcal{E}) = \textstyle\sum_i c_i(\mathcal{E}), \quad ch(\mathcal{E}) = \sum_{1 \le i \le r} \exp(x_i), \quad Td(\mathcal{E}) = \prod_{1 \le i \le r} Q(x_i),$$

where $Q(x) = x/(1 - \exp(-x))$ and the formal Chern roots x_i satisfy

$$\textstyle\prod_{1 \le i \le r} (1 + x_i t) = \sum_i c_i(\mathcal{E}) t^i.$$

(b) The total Chern class and the Todd class of X are defined by

$$c(X) = c(T_X), \quad Td(X) = Td(T_X).$$

(c) By the Hirzebruch-Riemann-Roch theorem [19] we have

$$\chi(\mathcal{E}) = \int_X ch(\mathcal{E})\, Td(X). \tag{5.1.1}$$

We will need the following properties of the characteristic classes:

(d) For a short exact sequence of vector bundles $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, we have

$$c(\mathcal{E}) = c(\mathcal{E}')\, c(\mathcal{E}''), \quad ch(\mathcal{E}) = ch(\mathcal{E}') + ch(\mathcal{E}''), \quad Td(\mathcal{E}) = Td(\mathcal{E}')\, Td(\mathcal{E}''). \tag{5.1.2}$$

(e) For the tensor product of two vector bundles $\mathcal{E}, \mathcal{F}$ we have

$$ch(\mathcal{E} \otimes \mathcal{F}) = ch(\mathcal{E})\, ch(\mathcal{F}). \tag{5.1.3}$$

(f) For the exterior product we have

$$\textstyle\sum_i c_i\big(\bigwedge^p \mathcal{E}\big) t^i = \prod_{i_1 < \cdots < i_p} (1 + (x_{i_1} + \cdots x_{i_p})t). \tag{5.1.4}$$

(g) For the dual vector bundle $\mathcal{E}^\vee$, we have

$$c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E}). \tag{5.1.5}$$

5.2. Remarks. (i) The function $Q(x)$ has the expansion

$$Q(x) = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} x^{2k}, \tag{5.2.1}$$

where the B_k are the Bernoulli numbers, see [17], Ex. 3.2.4. The first few terms of B_k are

$$\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \dots$$

(ii) Using $c_i = c_i(\mathcal{E})$ and $r = \operatorname{rank} \mathcal{E}$, we have the expansions (see [17], Ex. 3.2.3–4)

$$\begin{aligned} ch(\mathcal{E}) &= r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + c_3) + \cdots, \\ Td(\mathcal{E}) &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}(c_1c_2) + \cdots. \end{aligned} \tag{5.2.2}$$

(iii) By (5.1.2), $c(\mathcal{E}), ch(\mathcal{E}), Td(\mathcal{E})$ are extended to well-defined morphisms

$$c(\mathcal{E}), ch(\mathcal{E}), Td(\mathcal{E}) : K^0(X) \to H^\bullet(X, \mathbf{Q}), \tag{5.2.3}$$

where the source is the Grothendieck group of vector bundles on X. Note that the initial term of $ch(\mathcal{E})$ is the virtual rank of $\mathcal{E}$, and the latter does not appear in $c(\mathcal{E}), Td(\mathcal{E})$.

(iv) For $n = \dim X$ we have

$$\int_X c_n(X) = \chi(X, \mathbf{C}), \quad \int_X Td(X)_n = \chi(X, \mathcal{O}_X), \tag{5.2.4}$$

where $\chi(X, \mathbf{C})$ is the topological Euler characteristic of X. For the first assertion, see e.g. [17], Ex. 8.1.12. The second assertion follows from the Hirzebruch-Riemann-Roch theorem (5.1.1) applied to $\mathcal{E} = \mathcal{O}_X$ where $c_i(\mathcal{E}) = 0$ for $i > 0$.

5.3. Combinatorial description of the cohomology. Let D be an essential central hyperplane arrangement. In the notation of (1.1) we apply the construction in (2.1) to

$$\mathcal{S} := \mathcal{S}(D)^{\mathrm{nnc}},$$

and not to $\mathcal{S}(D)$ as in [4], [5]. (This simplifies some arguments in loc. cit. considerably.) Note that $\gamma(V) \geq 2$ for $V \in \mathcal{S}$. By C. De Concini and C. Procesi [8] the cohomology ring of $\widetilde{Y}$ in (2.1) is described by using only the combinatorial data as follows:

Let $\mathbf{Q}[e_V]_{V \in \mathcal{S}}$ be the polynomial ring with independent variables e_V for $V \in \mathcal{S}$. There is an isomorphism

$$\mathbf{Q}[e_V]_{V \in \mathcal{S}} / I_{\mathcal{S}} \xrightarrow{\sim} H^\bullet(\widetilde{Y}, \mathbf{Q}), \tag{5.3.1}$$

sending e_V to $[E_V]$ for $V \neq 0$ and e_0 to $-[E_0]$, where E_0 is the total (or proper) transform of a general hyperplane which was denoted by $\widetilde{H}$. Moreover, the ideal $I_{\mathcal{S}}$ is generated by

$$R_{V,W} = \begin{cases} e_V e_W & \text{if } V, W \text{ are incomparable,} \\ e_V \widetilde{e}_W^{\,\gamma(W)-\gamma(V)} & \text{if } W \subsetneqq V, \\ \widetilde{e}_W^{\,\gamma(W)} & \text{if } V = \mathbf{C}^n, \end{cases} \tag{5.3.2}$$

where $\widetilde{e}_W := \sum_{W' \subset W} e_{W'}$ and $\gamma(V) := \operatorname{codim} V$. Here $V, W, W' \in \mathcal{S}$ except for the third case where $V = \mathbf{C}^n$. Note that $\mathcal{S}$ is stable by intersection so that a *nested* subset of $\mathcal{S}$ in the sense of [8] is always linearly ordered by the inclusion relation.

For $V \in \mathcal{S}(D) \setminus \mathcal{S}(D)^{\mathrm{nnc}}$, let $\mathbf{P}(V)^\sim$ denote the proper transform of $\mathbf{P}(V)$ in $\widetilde{Y}$. (Here the notation $\mathbf{P}(V)^{\mathcal{S}^V}$ in Section 2 cannot be used since $V \notin \mathcal{S} := \mathcal{S}(D)^{\mathrm{nnc}}$.) Then the cohomology class e_V of $\mathbf{P}(V)^\sim$ is given by $\prod_{D_j \supset V} e_{D_j}$ since $\mathbf{P}(V)^\sim$ is the intersection of $\mathbf{P}(D_j)^\sim$ with $D_j \supset V$. For $V = D_j$, we have by calculating the total transform of D_j

$$e_{D_j} + \textstyle\sum_{W \in \mathcal{S}, W \subset D_j} e_W = 0, \tag{5.3.3}$$

since e_0 corresponds to $-\widetilde{H}$. (This is similar to (5.3.2) for $\mathcal{S} = \mathcal{S}(D)$ although $\widetilde{Y}$ is different.)

5.4. Calculation of Chern classes. In our case the Chern classes of $\widetilde{Y}$ are expressed by applying inductively the formula for the Chern classes under the blow-up ([17], 15.4). By [4] we have $c(\widetilde{Y}) = \prod_{V \in \mathcal{S}} F_V$, under the isomorphism (5.3.1), with

$$F_V = \begin{cases} (1 - \widetilde{e}'_V)^{-\gamma(V)}(1 + e_V)(1 - \widetilde{e}_V)^{\gamma(V)} & \text{if } V \neq 0, \\ (1 - e_0)^n & \text{if } V = 0, \end{cases} \tag{5.4.1}$$

where $\widetilde{e}_V := \sum_{W\subset V} e_W$, $\widetilde{e}'_V := \widetilde{e}_V - e_V$, and $\gamma(V) := \operatorname{codim} V$. Using the Grothendieck group as in (5.2.3), (5.4.1) implies that $Td(\widetilde{Y}) = \prod_{V\in\mathcal{S}} G_V$ with

$$(5.4.2)\qquad G_V = \begin{cases} Q(-\widetilde{e}'_V)^{-\gamma(V)} Q(e_V) Q(-\widetilde{e}_V)^{\gamma(V)} & \text{if } V\ne 0, \\ Q(-e_0)^n & \text{if } V = 0, \end{cases}$$

So the Chern classes and the Todd class of $\widetilde{Y}$ are expressed by using only the combinatorial data via (5.3.1).

Set $\mathcal{S}' = \mathcal{S}(D)^{\mathrm{nnc}} \cup \{D_i\}$ where the D_i are the irreducible components of D. The proper transform $\mathbf{P}(D_i)^\sim$ of $\mathbf{P}(D_i)$ in $\widetilde{Y}$ will be denoted by E_{D_i} . By (5.3.3) its cohomology class e_{D_i} is given by

$$e_{D_i} = -\sum_{W\in\mathcal{S},\, W\subset D_i} e_W \in \mathbf{Q}[e_W]_{W\in\mathcal{S}}/I_{\mathcal{S}}.$$

We have a short exact sequence

$$0 \to \Omega^1_{\widetilde{Y}} \to \Omega^1_{\widetilde{Y}}(\log \widetilde{Z}) \to \bigoplus_{V\in\mathcal{S}'\setminus\{0\}} \mathcal{O}_{E_V} \to 0.$$

Using (5.2.3), we get then

$$(5.4.3)\qquad c(\Omega^1_{\widetilde{Y}}(\log \widetilde{Z})) = c(\Omega^1_{\widetilde{Y}}) \prod_V c(\mathcal{O}_{E_V}) = c(\Omega^1_{\widetilde{Y}}) \prod_V c(\mathcal{O}_{\widetilde{Y}}(-E_V))^{-1}.$$

Moreover, the Chern classes of $\Omega^p_{\widetilde{Y}}(\log \widetilde{Z}) = \bigwedge^p \Omega^1_{\widetilde{Y}}(\log \widetilde{Z})$ for $p > 1$ are expressed by using those of $\Omega^1_{\widetilde{Y}}(\log \widetilde{Z})$ by (5.1.4). (However, it is not easy to write down the universal polynomials explicitly.)

5.5. Proofs of Theorems 1–2 by HRR. We calculate the right-hand side of (1.5.1) in Proposition (1.5) by applying the Hirzebruch-Riemann-Roch theorem (5.1.1) to

$$(5.5.1)\qquad \mathcal{E}_k = \Omega^p_{\widetilde{Y}}(\log \widetilde{Z}) \otimes_{\mathcal{O}} \mathcal{O}_{\widetilde{Y}}(-k\widetilde{H} + \textstyle\sum_j \lfloor km_j/d \rfloor E_j),$$

where the m_j are given by $\mu(V)$ if E_j in Proposition (1.5) is E_V in (2.1). For $\mathcal{O}_{\widetilde{Y}}(D_k)$ with

$$D_k = -k\widetilde{H} + \textstyle\sum_j \lfloor km_j/d \rfloor E_j,$$

we have $c(\mathcal{O}_{\widetilde{Y}}(D_k)) = 1 + [D_k]$. Then we can apply (5.1.3) to calculate $ch(\mathcal{E}_k)$, and $\chi(\mathcal{E}_k)$ depends only on the combinatorial data using the assertions in (5.4) together with the Hirzebruch-Riemann-Roch theorem (5.1.1). In the reduced case the $n_{f,\alpha}$ depend only on the combinatorial data as in Theorem 2. Moreover, if $p = 0$ or $p = n-1$, then $n_{f,\alpha}$ for $\alpha \in (0,1] \cup (n-1,n)$ depends only on the weak equivalence class using (1.4.5) for $p = n-1$ since $\Omega^0_{\widetilde{Y}}(\log \widetilde{Z}) = \mathcal{O}_{\widetilde{Y}}$ for $p = 0$. So the assertion follows.

5.6. Remark. We can prove Theorem (4.4) by using (5.3–4), and this is enough for the proofs of Theorems 1–2 as is shown in (4.5).

In the following, we illustrate how to calculate $n_{f,\alpha}$ using this method.

5.7. Proof of Theorem 3 by HRR. Let a_i denote the $e_V \bmod I_{\mathcal{S}}$ in (5.3) for $V \in \mathcal{S}^{(2)}$ (see (1.1.3)), i.e. the a_i correspond to the points of $\mathbf{P}(D^{\mathrm{nnc}})$. Set $c = e_0$. We have the relations

$$a_i a_j = 0\,(i \ne j),\ a_i c = 0,\ a_i^2 = -c^2,\ c^3 = 0,$$

using $(a_i + c)^2 = 0$, etc. in (5.3.2). Let F'_i denote F_V for V corresponding to a_i. Then

$$F'_i = (1-c)^{-2}(1+a_i)(1-c-a_i)^2 = 1 - a_i + c^2,$$

and $F_0 = (1-c)^3$. Set $\nu^{(2)} = \sum_{m\ge 3} \nu_m^{(2)}$ with $\nu_m^{(2)}$ as in (1.1.3). Since $c(\widetilde{Y}) = F_0 \prod_i F'_i$, we get

$$c(\widetilde{Y}) = 1 - (\textstyle\sum a_i + 3c) + (\nu^{(2)} + 3)c^2, \quad Td(\widetilde{Y}) = 1 - \dfrac{1}{2}(\textstyle\sum a_i + 3c) + c^2,$$

using (5.2.2). Note that $\Omega^2_{\widetilde{Y}} = \mathcal{O}_{\widetilde{Y}}(-3\widetilde{H} + \sum_i E_i)$ where the E_i correspond to a_i. So the Hirzebruch-Riemann-Roch theorem for a line bundle coincides with the Riemann-Roch theorem for surfaces, and the argument is the same as in (3.1) if $p = 2$ or 0.

In (3.1), the assertion for $p = 1$ is reduced to the other cases using the relation with $\chi(U)$. However, it should be possible to prove it by using the Hirzebruch-Riemann-Roch theorem for vector bundles. We apply this to

$$\mathcal{E}_k = \Omega^1_{\widetilde{Y}}(\log \widetilde{Z}) \otimes \mathcal{O}_{\widetilde{Y}}(D_k),$$

with $D_k = -k\widetilde{H} + \sum_j \lfloor km_j/d \rfloor E_j$. By the calculation of $c(\widetilde{Y})$ together with (5.1.5) we have

$$c(\Omega^1_{\widetilde{Y}}) = 1 + \textstyle\sum a_i + 3c + (\nu^{(2)} + 3)c^2.$$

So we get by (5.4.3)

$$c(\Omega^1_{\widetilde{Y}}(\log \widetilde{Z})) = (1 + \textstyle\sum a_i + 3c + (\nu^{(2)} + 3)c^2) \prod_i (1 - a_i)^{-1} \prod_j (1 - a'_j)^{-1}.$$

Here $a'_j := -\sum_{i\subset j} a_i - c$ which corresponds to the proper transform of an irreducible component $\mathbf{P}(D_j)$ of $\mathbf{P}(D)$, and we write $i \subset j$ if the corresponding subspaces of $\mathbf{P}^2$ have such an inclusion relation. (Note that $-c$ corresponds to $\widetilde{H}$.) We have

$$\textstyle\prod_j (1 + \sum_{i\subset j} a_i + c) = 1 + \sum_i m_i a_i + dc + \big(\binom{d}{2} - \sum_i \binom{m_i}{2}\big)c^2,$$

and $c(\Omega^1_{\widetilde{Y}}(\log \widetilde{Z}))$ is equal to

$$1 + \textstyle\sum_i (2 - m_i)a_i + (3-d)c - \frac{1}{2}\big((d^2 - 5d + 2\nu^{(2)} + 6) - \sum_i (m_i^2 - 3m_i + 4)\big)c^2.$$

Then $Td(\widetilde{Y})$, $\frac{1}{2}ch(\Omega^1_{\widetilde{Y}}(\log \widetilde{Z}))$, $ch(\mathcal{O}_{\widetilde{Y}}(D_k))$ are respectively

$$\begin{aligned}
&1 - \tfrac{1}{2}(\textstyle\sum_i a_i + 3c) + c^2,\\
&1 - \tfrac{1}{2}\big(\textstyle\sum_i (m_i - 2)a_i + (d-3)c\big) + \tfrac{1}{4}\big(\textstyle\sum_i m_i - d - 2\nu^{(2)} + 3\big)c^2,\\
&1 + \big(\textstyle\sum_i \lfloor km_i/d \rfloor a_i + kc\big) - \tfrac{1}{2}\big(\textstyle\sum_i \lfloor km_i/d \rfloor^2 - k^2\big)c^2.
\end{aligned}$$

Calculating the degree 2 part of the multiplication of these three, we get the right-hand side of the second equation divided by -2 in Theorem 3 where $i = d - k$.

5.8. Proof of Theorem 4 by HRR. Let $a_j\,(j \in I')$, $b_k\,(k \in I'')$, c denote the e_V mod $I_{\mathcal{S}}$ in (5.3) for $V \in \mathcal{S}^{(i)}$ (see (1.1.3)) depending on whether $i = 2$ or 3 or 4. We will write $k \subset j$ if there is an inclusion relation between the corresponding V. Let $n_j\,(j \in I')$ be the number of $b_k\,(k \in I'')$ with $k \subset j$. We have the relations

$$\begin{aligned}&a_i a_j = b_k b_l = a_j b_k^2 = a_j c^2 = b_k c = 0\,(i \ne j,\, k \ne l),\; a_j b_k = 0\,(k \not\subset j),\\ &a_j b_k = -a_j c\,(k \subset j),\; a_j^3 = -2(n_j - 1)c^3,\; a_j^2 c = b_k^3 = -c^3,\; a_j^2 b_k = c^3\,(k \subset j),\end{aligned}$$

using $(a_j + \sum_{k \subset j} b_k + c)^2 = 0$, $(b_k + c)^3 = 0$, $a_j(b_k + c) = 0\,(k \subset j)$, $a_j c^2 = 0$, see (5.3.2). By the same argument as in (4.8) it is enough to calculate $\chi(\mathcal{E})$ for a line bundle $\mathcal{E}$ with

$$c_1(\mathcal{E}) = \textstyle\sum_j A_j a_j + \sum_k B_k b_k + Cc \;\; (A_j, B_k, C \in \mathbf{Z}). \tag{5.8.1}$$

Let F'_j, F''_k denote F_V if V corresponds to a_j, b_k respectively. Then

$$F'_j = 1 - a_j - a_j^2 + 2(n_j - 1)a_j c - 2(n_j - 1)c^3, \quad F''_k = 1 - 2b_k - 2c^3.$$

Since $c(\widetilde{Y}) = F_0 \prod_j F'_j \prod_k F''_k$ with $F_0 = (1 - c)^4$, we get

$$c_1(\widetilde{Y}) = -\textstyle\sum_j a_j - \sum_k 2b_k - 4c, \quad c_2(\widetilde{Y}) = \sum_j (2a_j c - a_j^2) + 6c^2,$$

where $c_3(\widetilde{Y})$ is the topological Euler characteristic $\chi(\widetilde{Y})$ multiplied by $-c^3$, see (5.2.4). This gives $Td(\widetilde{Y})$ using the expansion of the Todd class in (5.2.2) (where c_3 does not appear so that $c_3(\widetilde{Y})$ is not needed). Thus we get

$$\begin{aligned}Td(\widetilde{Y})_1 &= -\textstyle\sum_j \dfrac{a_j}{2} - \sum_k b_k - 2c,\\ Td(\widetilde{Y})_2 &= \textstyle\sum_j \dfrac{(5 - 2n_j)a_j c}{6} + \sum_k \dfrac{b_k^2}{3} + \dfrac{11c^2}{6},\end{aligned}$$

where $Td(\widetilde{Y})_3 = -c^3$, see (5.2.4). We apply this theorem to the line bundle $\mathcal{E}$ in (5.8.1). Then we get

$$\chi(\mathcal{E}) = \textstyle\sum_j \big(2(n_j - 1)\binom{A_j+1}{3} - \binom{A_j}{2}\big(\sum_{k \subset j} B_k - C + 1\big)\big) + \sum_k \binom{B_k}{3} - \binom{C-1}{3}, \tag{5.8.2}$$

which is compatible with (4.8.1) where C corresponds to $-C$. We apply this to $\mathcal{E}$ with $A_j = 2 - \lceil im_j/d \rceil$, $B_j = 3 - \lceil im_j/d \rceil$, $C = 4 - i$, where $i = d - k$ and $p = 3$ in Proposition (1.5), see (1.4.5). Then Theorem 4 follows.

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